A Partial Order on Preference Profiles*

Wayne Yuan Gao†

August 18, 2021

Abstract

We propose a theoretical framework under which preference profiles can be meaningfully compared. Specifically, given a finite set of feasible allocations and a preference profile, we first define a ranking vector of an allocation as the vector of all individuals' rankings of this allocation. We then define a partial order preference profiles and write "$P \trianglerighteq P'$", if there exists an onto mapping ψ from the Pareto frontier of P' onto the Pareto frontier of P, such that the ranking vector of any Pareto efficient allocation under P' is weakly dominated by the ranking vector of the image allocation under ψ under P. We provide a characterization of the maximal and minimal elements under the partial order. In particular, we illustrate how an *individualistic* form of social preferences can be $\trianglerighteq$-maximal in a specific setting. We also discuss how the framework can be further generalized to incorporate additional economic ingredients.

Keywords: preference profile, partial order, Pareto efficiency, comparative statics, ranking

*I thank Yi Chen, Shaowei Ke, George Mailath, Xiaosheng Mu, Andrew Postlewaite, Larry Samuelson and Chen Zhao for helpful discussions, comments and suggestions. I also thank Simon Oros for capable research assistance.

†Gao: Department of Economics, University of Pennsylvania. 133 South 36th Street, Philadelphia, PA 19104, USA, waynegao@upenn.edu.

1 Introduction

Preferences, and preference profiles as the collection of the preferences held by a set of individuals, are often times the primitives of an economic model, and they also usually form the basis for the evaluation of economic outcomes.

However, in real life, preferences themselves are often discussed and compared; furthermore, sometimes people may even hold strong opinions on such comparisons. For an example of considerable social importance, consider the advance of LGBT rights, along with the accompanying public debates and evolving social attitudes towards the LGBT community (Belmonte, 2020).

In addition, even though preferences are typically taken to be exogenously given in most economic analysis, at least to some degree preferences can be shaped within a society through a variety of social environments, such as families, schools, companies and governments: see Postlewaite (2011) a survey on the social determinants of preferences. A natural question to ask then is: if an individual, an entity or the society as a whole can choose what kind of preferences (and consequently preference profiles) to induce in a target population, then in what formal sense can one say that a certain type of preference (profile) is "better than" an alternative type on a normative ground?

While *by no means* this paper can capture the sophistication of these complex but important social topics, we seek to propose a theoretical framework for a potentially meaningful comparison of preference profiles, though admittedly on a very stylized and abstract level.

Specifically, we propose a partial order on preference profiles. Given a fixed (finite) set of feasible allocations X, and any preference profile P on X, we first define the Pareto frontier $PE_X(P)$ under P and X. For each Pareto efficient allocation $x \in PE_X(P)$, we then define each individual i's ranking $R_X(P_i, x)$ of x among all allocations in X under P_i. We then define a partial order on preference profiles based on the ranking evaluations of Pareto frontiers. We say one preference profile P is "*more Pareto-favorable than*" another preference P', and write "$P \trianglerighteq_X P'$", if there exists an *onto* mapping ψ from the Pareto frontier of P' onto the Pareto frontier of P, such that the ranking vector of any Pareto efficient allocation under P' is weakly dominated by the ranking vector of the image allocation under ψ :

$$R_X(P, \psi(x)) \leq R_X\left(P', x\right), \quad \forall x \in PE_X\left(P'\right).$$

We now provide a discussion about this proposed partial order, explain why it may be sensible, and discuss its difference from related literature.

First, the proposed partial order on preference profiles is defined purely based on *intra-*

personal comparisons of a given individual's own ranking evaluations according to two candidate preferences of her own. It should be emphasized that there is no whatsoever *interpersonal* comparison or aggregation of preferences, and hence this paper is *not* an analysis of social welfare, social justice or social equality, which have been extensively studied in welfare economics, dating back to the heated debates in the pioneering work by Rawls (1958), Harsanyi (1975) and Binmore (1989). Conceptually, welfare economics is primarily concerned with the selection of some allocations, or in certain scenarios a single allocation, from the set of all feasible allocations, under a *single* given preference profile. This paper adopts Pareto efficiency as such a solution concept, but is primarily concerned with comparisons across *different* preference profiles.

Second, the proposed partial order on preference profiles does not reflect the "*preference*" of any external entities, but only encodes the preferences of the set of involved individuals themselves. Under a "weakly more Pareto-favorable" preference profile, everyone may obtain a weakly higher-ranked allocation according to their own preferences on the Pareto frontier, relative to what they may obtain in the Pareto frontier of a "weakly less Pareto-favorable" preference profile. The weak ranking improvement holds for every individual in the society.

Third, the comparison across different preference profiles via the proposed partial order is anchored by two invariant objects: the fixed set of feasible allocations X, and the *ordinal* structure of preferences on X. For a given individual, two distinct preferences of this individual are specified on exactly the same set of objects, and given preferences are ordinal, all preferences are completely characterized by the relative ranking of different allocations in X. Suppose that we *have to* make a systematic comparison across different preference profiles, then we have to rely on *some i*nvariant structures of preference profiles as a lever of comparison, and in this case the use of ranking seems at least natural, if not necessary under some axiomatic foundations.

Forth, conceptually the analysis in this paper can be thought as a study of monotone comparative statics in an inverted direction: we induce a partial order on preference profiles based on a partial order on the set of Pareto efficient allocations, and then seek to characterize the extremal elements (as well as intermediate transitions) under the induced partial order. The techniques underlying the analysis in this paper is combinatorial in nature, and is in particular based on the standard approach for monotone comparative statics that exploit lattice theory and supermodularity (Topkis, 1998; Milgrom and Shannon, 1994).

Lastly, as should be clear from the discussion above, this paper does not move outside the paradigm of *economic rationality*, but only explores a potential framework that compares preference profiles, taking as given the economic outcomes that may arise from each given preference profile. In this paper, we use Pareto efficiency as a solution concept for economic outcomes, but other solution concepts based on efficiency or equilibrium considerations may be of interest in different scenarios. The individuals in this paper has full rationality with respect to their common knowledge of the preference profile, the associated Pareto frontier and their optimal choices. Hence, this paper itself does not fall into, or seek to address, the debate between rationalistic economics and behavioral economics or neuroeconomics, represented by Gul and Pesendorfer (2008), Camerer (2008) and Hausman (2008) in Caplin and Schotter (2008).

The rest of the paper is organized as following. Section 2 illustrates the key idea underlying the proposed partial order via a simple two-individual two-allocation example. Section 3 lays out the model setup in a general finite allocation space with general preferences, and characterize the maximal and minimal elements under the proposed partial order in the space of all possible preference profiles. Section 4.1 considers a setting where both the allocation space and the preference profiles are endowed with some sensible structures: specifically, we consider the assignment of one and only one widget among a discrete set of widgets to each individual, where each individual's preference is private in the sense that she only cares about her own widget assignment. Section 4.1 contains an illustrative example with continuous allocation space, and we conclude in Section 6 with a discussion about how the framework can be further generalized to incorporate additional economically relevant modeling ingredients. The proofs are available in the Appendix.

2 A Two-Individual Two-Allocation Example

We start with a simple example with two individuals and two feasible allocations to illustrate the key idea of this paper.

Specifically, let "1" and "2" denote two individuals, and "r", "b" denote two pens: "r" for a red pen and "b" for a blue pen. Consider the problem of allocating the two pens to the two individuals, under the assumption that each of the two individuals must be allocated one and only pen. Then, the set of admissible allocations, denoted by X, may be written denoted as

$$X := \{(r, b), (b, r)\},$$

where "(r, b)" stands for "allocate the red pen to individual 1, and the blue pen to individual 2", and "(b, r)" the vice versa.

Consider two possible strict ordinal preference profiles $P = (P_1, P_2)$ and $P' = \left(P'_1, P'_2\right)$ on the set of admissible allocations X, where "P_i" and "P'_i" stand for individual i's preferences.

Suppose that under the first preference profile P, individual 1 prefers the red pen over the blue pen, while individual 2 prefers the blue pen over the red pen (and neither individual intrinsically cares about what pen the other individual is getting), inducing a preference profile on X defined by:

$$P: \quad (r,b) \succ_{P_1} (b,r), \quad (r,b) \succ_{P_2} (b,r), \tag{1}$$

where "$\succ_{P_i}$" denotes strict preference according to P_i. Under the second preference profile $P^{'}$, suppose that both individual 1 and individual 2 prefer getting the red pen over the blue pen:

$$P^{'}: \quad (r,b) \succ_{P_1^{'}} (b,r), \quad (b,r) \succ_{P_2^{'}} (r,b). \tag{2}$$

The question we ask is: is there any *precise* and *sensible* way to compare the two preference profiles P and $P^{'}$?

For this simple example, it is straightforward to see that, under the first preference profile P, there is a unique Pareto efficient allocation (r,b), which allocates to each individual her favorite pen. However, under the second preference profile $P^{'}$, both allocations (r,b) and (b,r) in X are Pareto efficient, and moreover there is at least one individual who does not get her favorite pen. It is then at least *intuitive* to say that the first preference profile P is "better" in the sense that individuals are able to obtain their best possible allocation, according to each of their own preferences in P, via *market exchanges*, while the same cannot be said for the second preference profile $P^{'}$.

To represent the idea above mathematically, we first compute the sets of Pareto efficient allocations in X under P and $P^{'}$, denoted by $PE_X(P)$ and $PE_X\left(P^{'}\right)$:

$$PE_X(P) = \{(r,b)\}, \quad PE_X\left(P^{'}\right) = \{(r,b),(b,r)\}.$$

Then, for each preference profile, we compute each individual's *ranking* of all Pareto efficient allocations in X according to this given individual's preference. Specifically, under P, we have only one Pareto efficient allocation (r,b), to which individual 1 assigns a ranking of $R_X(P_1,(r,b)) = 1$ according to individual 1's preference P_1 on X, and to which individual 2 also assigns a ranking of $R_X(P_2,(r,b)) = 1$, or in vector form:

$$R_X(P,(r,b)) = (1,1).$$

Similarly, under $P^{'}$, we compute the ranking vectors of the two Pareto efficient allocations in $PE_X\left(P^{'}\right)$ as

$$R_X\left(P^{'},(r,b)\right)=(1,2)\,,\quad R_X\left(P^{'},(b,r)\right)=(2,1)\,.$$

In this example, the intuitive urge, if any, to consider P as "*better than*" $P^{'}$, should be coming from the observation that all Pareto efficient allocations under P are assigned *better ranking vectors* than all Pareto efficient allocations under $P^{'}$:

$$R_X\left(P,x\right)\lneqq R_X\left(P^{'},y\right),\ \forall x\in PE_X\left(P\right),\ \forall y\in PE_X\left(P^{'}\right). \tag{3}$$

We may define "$P\rhd_X P^{'}$" based on (3).

We offer some immediate remarks. The first is on the comparison of rankings across different preference profiles.

Remark 1 (About the Use of Ranking)*.* Clearly, the definition of the comparison "$P\rhd_X P^{'}$" relies crucially on the comparison of the ranking vectors across allocations in P and $P^{'}$ based on (3), and whether such a comparison is meaningful or not at all is admittedly not a trivial question.

However, we argue that, at for this simple two-individual two-allocation example, the proposed comparison "$P\rhd_X P^{'}$" has some nice features. First, we have not pre-imposed any form of preference relations on the allocation set X from outside the two given preference profiles P and $P^{'}$. Second, we have maintained completely symmetric treatment of individuals beyond their potentially heterogeneous preferences on X. Third, given the previous two points, if we have to make a comparison between preference profiles, it seems very hard to imagine a scenario where a preference profile like P, under which there is a unique reasonable economic outcome allocation (the single Pareto efficient allocation) that every individual finds to be her own unique favorite allocation among the set of all admissible allocations X, is not defined to the "best possible" preference profile.

Admittedly, these are not precisely defined axioms that necessitate the use of rankings as the "invariant lever of comparison" across different preference profiles. However, it is hoped that this remark provides some intuitive motivations for the use of rankings, and it seems interesting for future research to explore the axiomatic approaches that either support the use of rankings or suggest new definitions.

Remark 2 (About Pareto Dominance)*.* It should be pointed out that the comparison between two preference profiles P and $P^{'}$ described above is not a comparison about *social welfare* per se. Recall that, in the standard economics literature, a welfare comparison is made between two admissible (feasible) allocations in the set of all admissible (feasible) allocations X, under a *single* given preference profile P. The comparison "$P \rhd_X P^{'}$", however, involve two preference profiles P and $P^{'}$. On the other hand, the comparison "$P \rhd_X P^{'}$" obviously has tight relationship with *Pareto dominance,* an important concept in welfare economics.

First, the comparison "$P \rhd_X P^{'}$" is based on the Pareto efficient allocations only under each preference profile. The focus on the Pareto efficient allocations is intended to impute, or at least to represent, economic gains from any trade or exchange opportunities under each given ordinal preference profile. On a more abstract, the set of Pareto efficient allocations, or the mapping PE_X, represent one form of well-defined *economic solution concepts* defined for any given preference profile. In particular, the solution concept of Pareto efficiency is based on a specific form of efficiency criterion. More generally, one could substitute Pareto efficiency PE_X with any economic solution concept $\mathcal{E}_X$ that picks a set of *economic outcome* allocations within X. It is popular to select $\mathcal{E}_X$ either based on *efficiency* considerations or *equilibrium* considerations in economics. In this paper, we adopt Pareto efficiency as a representative solution concept, not only because that it is arguably the most robust form of efficiency criteria, but also because that, relatedly, in many economic scenarios the corresponding equilibrium outcomes are also Pareto efficient.

Second, the comparison "$P \rhd_X P^{'}$" based on 6 is based on uniform (elementwise) inequality between the ranking vectors taken from the two sets of Pareto efficient allocations under two different preference profiles, which shares some formal similarity with the definition of Pareto dominance. This is intended to make the comparison "$P \rhd_X P^{'}$" as robust as possible. In particular, notice that in the description leading up to 6, we have not made any comparison (or aggregation) of preference relations *across* different individuals: every individual's ranking of each admissible allocation is either compared with her own ranking of another admissible allocation under the same preference profile, or compared with her own ranking of some allocation under a different preference profile.

3 General Preferences on General Allocation Space

3.1 Setup and Definition

Let $N := \{1, ..., N\}$ be a set of N individuals (with slight abuse of notation), and let $X := \{1, ..., M\}$ be a set of M admissible or feasible allocations (or actions). Let P_i denote each individual i's preference on X, with "$x \succ_{P_i} x^{'}$" denoting "i strictly prefers x over $x^{'}$ according to P_i", and "$x \succsim_{P_i} x^{'}$" and "$x \sim_{P_i} x^{'}$" denoting weak preference of x over $x^{'}$

and indifference between x and $x^{'}$ according to P_i, respectively. Let $P := (P_i)_{i \in N}$ for the preference profile. We say a preference profile P is *strict* if either $x \succ_{P_i} x^{'}$ or $x^{'} \succ_{P_i} x$ holds for any $i \in N$ and any two distinct $x, x' \in X$.

Given the feasibility set X and a given individual i's preference P_i on X, denote individual i's ranking of a given allocation $x \in X$ among all allocations in X according to i's own preference P_i as

$$R_X(P_i, x) := 1 + \#\{z \in X : z \succ_{P_i} x\}, \ \forall x \in X,$$

i.e., x is the $R_X(P_i, x)$-th best allocation in X according to i's preference P_i. We may write the ranking vector of a given allocation x under P as

$$R_X(P, x) := (R_X(P_1, x), R_X(P_2, x), ..., R_X(P_N, x)).$$

Given the feasibility set X and the preference profile P on X, denote the set of Pareto efficient allocations in X under the preference profile P by

$$PE_X(P) := \{x \in X : \ x \text{ is not strictly Pareto dominated by any } z \in X\}.$$

We now propose the following definition of a partial order on preference profiles, which generalizes the one described in Section 2 via 3. An important and necessary departure from the two-individual two-allocation case is to accommodate the fact that, in general, the ranking vector of an arbitrary Pareto efficient allocation under P may not be comparable to an arbitrary Pareto efficient allocation under $P^{'}$. Our proposed adaption is to define the partial order by the existence, or lack thereof, of a mapping from $PE\left(P^{'}\right)$ to $PE(P)$ such that the ranking vectors are comparable along the mapping in a consistent way.

Definition 1 (A Partial Order $\trianglerighteq_X$ on Preference Profiles)**.** Given a feasibility set X and any two preference profiles P and $P^{'}$ defined on X, we say that P is weakly "*more Pareto-favorable than*" $P^{'}$ on X, and write

$$P \trianglerighteq_X P^{'},$$

whenever there exists an *onto* mapping $\psi : PE_X\left(P^{'}\right) \to PE_X(P)$ such that

$$R_X(P, \psi(x)) \leq R_X\left(P^{'}, x\right), \quad \text{for any } x \in PE_X\left(P^{'}\right).$$

We write "$P \vartriangleright_X P^{'}$" for the strict case, where $P \trianglerighteq_X P^{'}$ but $P^{'} \ntrianglerighteq_X P$. We write "$P \triangleq_X P^{'}$" for the case where $P \trianglerighteq_X P^{'}$ and $P^{'} \trianglerighteq_X P$.

Clearly, $\trianglerighteq_X$ is a partial order. The requirement of ψ being *onto* is to ensure that there is no allocation $y \in PE_X(P)$ that does not dominate some $x \in PE_X\left(P'\right)$ in terms of the ranking vector. It is easy to check that, for the two-individual two-allocation example in Section 2, we may set the mapping ψ by $\psi(x) = (r, b)$ for either $x \in PE_X\left(P'\right) = \{(r, b), (b, r)\}$.

For notational simplicity, in the following we will suppress the subscript "$\cdot_X$" in "R_X", "PE_X" and "$\trianglerighteq_X$", given that X is a primitive of the analysis in this paper. We will write out the subscript "$\cdot_X$" explicitly when we emphasize the dependence of the partial order on X.

There are potentially many other ways to define the partial order as a generalization of the one proposed in the two-individual two-allocation example. We discuss in the following remarks, as well as the corresponding appendices, two alternative versions of definitions that are different from but closely related to Definition 1.

Remark 3. We may define an alternative version $\tilde{\trianglerighteq}$ of the partial order in the following way. We define a quasi order $P \tilde{\triangleright} P'$, if there exists an *onto* mapping $\psi : PE\left(P'\right) \to PE(P)$ such that

$$R(P, \psi(x)) \leq R\left(P', x\right), \quad \text{for any } x \in PE\left(P'\right),$$

with at least one inequality being strict in the following sense:

$$R(P_i, \psi(x)) < R\left(P_i', x\right), \quad \text{for some } i \in N \text{ and some } x \in PE\left(P'\right).$$

Note that $\tilde{\triangleright}$ and $\triangleright$, though very similar, are not equivalent in general. In particular, note that $\tilde{\triangleright}$ is stronger than $\triangleright$ in the sense that $P \tilde{\triangleright} P'$ implies $P \triangleright P'$ but not vice versa. See Appendix for more discussion and adapted results on $\tilde{\triangleright}$.

Remark 4. We may define an alternative version $\hat{\trianglerighteq}_X$ of the partial order in the following way. We define $P \hat{\trianglerighteq} P'$, if the following two conditions hold simultaneously:

(i) For any $x \in PE\left(P'\right)$, there exists some $y \in PE(P)$ such that $R(P, y) \leq R\left(P', x\right)$.

(ii) For any $y \in PE(P)$, there exists some $x \in PE\left(P'\right)$ such that $R(P, y) \leq R\left(P', x\right)$.

Note that $\hat{\trianglerighteq}$ is weaker than $\trianglerighteq$ in the sense that $P \trianglerighteq P'$ implies $P \hat{\trianglerighteq} P'$ but not vice versa. See Appendix for more discussion and adapted results on $\hat{\trianglerighteq}$.

In the main text, we will focus on Definition 1.

3.2 Extremal Elements

We now provide a characterization of the maximal and minimal elements under the partial order $\trianglerighteq$ on the space of all possible preference profiles on X.

We say that a preference profile P is a $\trianglerighteq$-*maximal element* if $P' \trianglerighteq P$ implies that $P' \triangleq P$, that P is a $\trianglerighteq$-*minimal element* if $P \trianglerighteq P'$ implies that $P \triangleq P'$, that P is a $\trianglerighteq$-*upper bound* if $P \trianglerighteq P'$ holds for any P', and that P is a $\trianglerighteq$-*lower bound* if $P' \trianglerighteq P$ holds for any P'.

Theorem 1 (Extremal Elements under $\trianglerighteq$)**.** *Consider the space of all possible preference profiles on X.*

(a) Maximal elements and least upper bounds: $\overline{P}$ is a $\trianglerighteq$-maximal element and a $\trianglerighteq$-upper bound if and only if there exists a unique $x^* \in X$ such that all individuals like x^* the best under $\overline{P}$, i.e.,

$$R\left(\overline{P}\right)[x^*] = \mathbf{1}_N. \tag{4}$$

(b) Minimal elements: $\underline{P}$ is a $\trianglerighteq$-minimal element if and only if both of the following conditions hold:

(i) every allocation in X is Pareto efficient under $\underline{P}$, i.e.,

$$PE\left(\underline{P}\right) = X. \tag{5}$$

(ii) $\underline{P}$ is strict.

Lower bounds do not exist in general. Example:

$$P = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}, \quad P' \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

P and P' are both minimal elements as $PE\left(P\right) = PE\left(P'\right) = X$. However, no preference profile P'' can be weakly dominated by both, $P \trianglerighteq P''$ and $P' \trianglerighteq P''$, because at least one dominance must be strict, which is impossible given the minimality of P and P'.

Theorem 1(a) is a direct generalization of the intuition in Remark 1 for the two-individual two-allocation example that, if everyone gets her unique favorite allocation in the unique Pareto efficient allocation under a given preference profile, then this preference profile should be considered as the "best". The proof is almost trivial: there exists a trivial onto mapping ψ from $PE\left(P\right)$ to $PE\left(\overline{P}\right) = \{x^*\}$ that satisfies Definition 1, given that $R\left(\overline{P}, x^*\right) = \mathbf{1}_n$

achieves the best possible ranking vector. Note that Theorem 1(a) does not say that $\overline{P}$ must be unique: in general there exist many $\trianglerighteq_X$-maximal preference profiles that are equivalent with each other under $\trianglerighteq_X$, and all such preference profiles are also $\trianglerighteq_X$-upper bounds.

Theorem 1(b), which characterizes the $\trianglerighteq_X$-minimal elements, is also a direct generalization of the preference profile $P^{'}$ defined in the two-individual two-allocation example, where $PE\left(P^{'}\right) = \{(r,b),(b,r)\} = X$. Intuitively, more strictly conflicting preference relations on X produce a larger set of Pareto efficient allocations, which would make it harder for a given allocation to Pareto dominate another , which requires congruent preference between two allocations across all individuals with at least one strict preference. Moreover, as more preference relations become strict, ranking vectors tend to worsen: with indifference, it might be possible to have two different allocations $x, x^{'}$ that some individual i finds to both be i's favorite allocation under some P_i, so that $R(P_i, x) = R\left(P_i, x^{'}\right) = 1$; however, breaking the indifference between x and $x^{'}$ in P_i, while keeping i's all other preferences on X unchanged, would increase the ranking vector as it is no longer possible to have $R(P_i, x) = R\left(P_i, x^{'}\right) = 1$.

Technically, the proof of Theorem 1(b) is involved, especially the "*only if*" part (with the proof of the "*if*" part being relatively simple once we have established the "*only if*" part). Despite the technicality, the proof of the "*only if*" part is worth some more discussion, which is sketched through three lemmas presented in the next subsection. These lemmas not only serves to characterize the minimal elements under $\trianglerighteq_X$, but also reveals how any non-minimal preference profile P that does not satisfies the condition in Theorem 1(b) can be *locally perturbed* to construct an alternative preference $P^{'}$ such that $P \vartriangleright_X P^{'}$. This is informative on comparisons between P and $P^{'}$ even if neither P or $P^{'}$ is maximal nor minimal.

3.3 Intermediate Transitions

In this subsection, we sketch the proof the "*only if*" part of Theorem 1(b) via three lemmas. Each of the following lemmas is proved via the construction of an alternative preference profile $P^{'}$ by "locally perturbing" a given preference profile P that is $\trianglerighteq_X$-minimal, followed by a demonstration of $P \vartriangleright P^{'}$. In words, we show how any non-minimal preference profile can be transformed towards a $\trianglerighteq$-minimal preference profile step by step.

Lemma (1) below, along with its proof, demonstrates how a preference profile P cannot be $\trianglerighteq$-minimal if the set of Pareto efficient allocations under P does not equal the whole allocation space X.

Lemma 1 (Universal Pareto Efficiency for Minimality)**.** *If $PE(P) \neq X$, then P is not $\trianglerighteq$-minimal.*

The detailed proof is available in the Appendix, but here we briefly describe how to construct an alternative preference profile P' such that $P \rhd P'$ when $PE(P) \neq X$.

Specifically, we may take any Pareto dominated allocation $\underline{x} \in X \backslash PE(P)$, which is Pareto dominated by some $\overline{x} \in PE(P)$. Moreover, we show that we can always select the allocation $\overline{x}$ to be such that some individual $\overline{i} \in N$ strictly prefers $\overline{x}$ over $\underline{x}$, and moreover $\overline{x}$ is individual $\overline{i}'s$ favorite allocation among all Pareto efficient allocations that Pareto dominate $\underline{x}$.

We then construct another preference profile P' by switching individual $\overline{i}$'s rankings of $\underline{x}$ and $\overline{x}$ in P only, i.e., setting

$$R\left(P'_{\overline{i}}, \underline{x}\right) := R\left(P_{\overline{i}}, \overline{x}\right), \quad R\left(P'_{\overline{i}}, \overline{x}\right) := R\left(P_{\overline{i}}, \underline{x}\right),$$

while keeping individual $\overline{i}$'s rankings of all other allocations, as well as all other individuals' rankings of all allocations, completely unchanged from P. Then it can be shown that $P \rhd P'$.

Under the newly constructed preference profile P', the allocation $\underline{x}$, which is not Pareto efficient under the original preference profile P, becomes Pareto efficient: $\underline{x} \in PE\left(P'\right)$. Intuitively, notice that the ranking vector for $\underline{x}$ is weakly improved uniformly from P to P', so any allocation that does not Pareto dominate $\underline{x}$ under P cannot Pareto dominates $\underline{x}$ under P', either. Moreover, individual $\overline{i}$'s ranking vector for $\underline{x}$ is strictly improved to such an extent that individual $\overline{i}$ now ranks $\underline{x}$ better under P' than all other Pareto efficient allocations that Pareto dominate $\underline{x}$ under P. Hence, $\underline{x}$ becomes Pareto efficient under P'. For this new Pareto efficient allocation $\underline{x}$, we can set the function ψ in Definition 1 to map $\underline{x} \in PE\left(P'\right)$ to $\overline{x} \in PE(P)$, so that the ranking vector of $\underline{x}$ under P' must still be no better than the ranking vector of $\psi(\underline{x}) = \overline{x}$ under P, because the rankings of all individuals other than $\overline{i}$ have not changed. If $\overline{x}$ remains Pareto efficient, we could also set ψ to map $\overline{x}$ to itself, noting that the ranking vector of $\overline{x}$ strictly worsens from P' to P, due to individual $\overline{i}$'s switch of rankings between $\overline{x}$ and a strictly inferior $\underline{x}$ from P.

The above summarizes the key intuition for why $P \rhd P'$, but of course the set of Pareto efficient allocations may involve other changes that we have not discussed above. Moreover, we also need to make sure that the mapping ψ can be fully configured in an appropriate way. The formal proof in Appendix A.1 contains all these details.

Lemmas (2) and (3) below, along with their proofs, demonstrates how a preference profile P cannot be $\trianglerighteq$-minimal if the preference P is not strict in two specific manners.

Lemma 2 (Partially Strict Preferences for Minimality)**.** *If there exist any two distinct individuals $i, j \in N$ and any Pareto efficient allocation $x \in PE(P)$ such that $x \sim_{P_i} y$ and*

$x \sim_{P_j} y$ for some $y \neq x$, then P is not $\trianglerighteq$-minimal.

Lemma 3 (Strict Preferences for Minimality)**.** *If there exists a Pareto efficient allocation $x \in PE(P)$ such that $x \sim_{P_i} y$ for some individual i and some allocation $y \neq x$, then P is not $\trianglerighteq$-minimal.*

Essentially, Lemma 3 states that, for $\trianglerighteq$-minimality, there cannot be any indifferences in P that involve any Pareto efficient allocation. The restriction to comparisons involving Pareto efficient allocations is very intuitive, given that the partial order $\trianglerighteq$ is defined based on comparisons of (ranking vectors of) Pareto efficient allocations.

Lemma 2 is less interpretable than Lemma 3, but it is only stated as an intermediate step to prove Lemma 3. Even though the proof of Lemma 2 is rather tedious, the underlying idea is quite simple: if we have two individuals ij and two allocations x, y such that $x \sim_{P_i} y$ and $x \sim_{P_j} y$, we can perturb the preference profile P by "breaking the indifferences" for individuals i and j in *opposite directions*, i.e., setting P' to ensure $x \succ_{P'_i} y$ and $y \succ_{P'_j} x$. In the meanwhile, we can keep all other preference relations unchanged, so that this perturbation essentially only affects the ranking between x and y.[1] This perturbation should have increased the ranking vectors uniformly across all allocations and all individuals. Of course, there are many more technical subtleties beyond this simple intuition: see the proof in Appendix A.2 for details.

The proof of Lemma 3 is much simpler once Lemma 2 is proved: essentially, we can just "break the indifference" between x and y for individual i. See the proof in Appendix A.3 for details.

4 Preferences on Structured Allocation Space

The previous section considers general preference profiles on an arbitrary finite set of feasible allocation space. However, we may often be interested in some forms of allocations that have meaningful structures, under which certain preference profiles may be automatically excluded from considerations. This section is thus intended to provide an illustration on how the key idea underlying Definition 1 can be flexibly adapted to accommodate and exploit sensible primitive structures or restrictions built in the space of admissible allocations as well as the space of admissible preference profiles.

[1] Technically, there can be other allocations z such that $z \sim_{P_i} x$ or $z \sim_{P_j} x$. See the proof in Appendix A.2 for how such kind of allocations are handled.

4.1 Private Preferences

Consider the allocation of M distinct indivisible widgets to a group of N individuals, where each individual is assigned one and only one widget.[2] Formally, we write the set of widgets as $M := \{1, ..., M\}$, and the set of admissible allocations as

$$X := \{x \in M^n : \# \{i : x_i = k\} \leq 1, \ \forall k \in M\} .$$

Suppose that each individual's preference is *private* in the sense that each agent only cares about the widget assigned to herself. Formally, each individual i's *private* preference is characterized by a *strict* preference P_i on the set of widget M. Given any allocation $x \in X$, individual i's ranking of x under P_i is dependent on x_i only:

$$R_X (P_i, x) \equiv R_M (P_i, x_i) := 1 + \# \{k \in M : k \succ_{P_i} x_i\} .$$

Note that, even though P_i is a strict preference on M, individual i's implied preference on X is not strict: conditional on getting a given widget $x_i \in M$, individual i is indifferent across all allocations $z \in X$ such that $z_i = x_i$.

The definition of the partial order $\trianglerighteq$ is still given in Definition 1.

It should be clear that the specification described here is a more structured generalization of the two-individual two-pen example in Section 2. Notice in particular that each individual's preference is restricted to be private, and thus constrained to allow for indifferences in a structured way. Hence, Theorem 1, which do not apply directly, as in this section we would like to respect the structures laid out above and focus on make comparisons only among preference profiles that satisfy these structures.

Theorem 2 (Extremal Elements among Private Preferences in Structured Allocation Space)**.** *Suppose $N \leq M$.*

(a) Maximal elements: A preference profile $\overline{P}$ is $\trianglerighteq_X$-maximal if and only if $\overline{P}$ is such that there exists N distinct widgets $\overline{x}_1, ..., \overline{x}_n$ in M such that $R\left(\overline{P}_i, \overline{x}_i\right) = 1$ for all $i \in N$, i.e. everyone's favorite widget in M is different.

(b) Minimal elements: A preference profile P is $\trianglerighteq_X$-minimal if and only if $\underline{P}$ is such that there exists a subset M_N of M such that (i) all individuals rank the widgets in M_N to be no worse than their N-th best widget in M, i.e., $R(\underline{P}_i, z) \leq N$ for all $i \in N$ and all $z \in M_N$, and (ii) all individuals preferences restricted on M_n coincide, i.e., $\underline{P}_i|_{M_N} = \underline{P}_j\Big|_{M_N}$ for all $i, j \in N$.

[2] We may take one of the widget as a "null widget", but we do not consider this case for simplicity.)

(c) *Extremal elements are also bounds: All preference profiles* $\overline{P}$ *are* $\succ_X$*-upper bounds, and all preference profiles* $\underline{P}$ *are* $\succ_X$*-lower bounds.*

Under the current setting, Theorem 2 provides a formal foundation for the desirability of *diversity,* or *individual heterogeneity*, in private preferences. Specifically, $\trianglerighteq$-maximality is characterized by full diversity in the top choices across individual private preferences, while $\trianglerighteq$-minimality is characterized by full alignment of the top-N choices in individual private preferences.

These results are highly consistent with the results in Theorem 1, despite the dissimilarities in appearance. For maximality, Theorem 2(a) and Theorem 1(a) are equivalent after accounting for the additional structures in the current setting. For minimality, though it is in general no longer possible to find a preference profile P such that P is strict and $PE(P) = X$ in the current specification, the full conformity of private preferences on the common top-N widgets under $\underline{P}$ in Theorem 2(b) induces an effectively *strict* preference profile over the set of allocations consisting of the top-N widgets only, which also coincides with the set of Pareto efficient allocations under $\underline{P}$.

However, compared with Theorem 1 in the last section, Theorem 2 further exploits the imposed structure built into the allocation space and the space of admissible preference profiles, thus gaining more specificity. It is plausible that, in many economic scenarios, the allocation space can be factorized as a product space of individual-specific allocations subject to some budget constraints over total allocations, and moreover in many scenarios individuals are more concerned with their own widget allocation rather than what widgets other individuals get. At least in such scenarios, Theorem 2 is more relevant or interesting than Theorem 1.

The proof of Theorem 1 utilizes the following sequence of lemmas and corollaries, each of which is interpretable and intuitive.

Lemma 4 (No Cycles of Envy)**.** *For any* $x \in PE(P)$*, if there are a sequence of distinct individuals* $i_1, ..., i_k$ *such that* $x_{i_{h+1}} \succ_{P_{i_h}} x_{i_h}$*for every* $h = 1, ..., k-1$*, we must have* $x_{i_k} \succ_{P_{ik}} x_{i_1}$.

The proof is simple. Otherwise, an obvious trading cycle would constitute a strict Pareto improvement. Note that the above holds in particular for pairs, i.e., $x_j \succ_{P_i} x_i \Rightarrow x_j \succ_{P_j} x_i$.

Lemma 5 (No Better Widgets Available)**.** *For any* $x \in PE(P)$*, any* $i \in N$ *and any* k *such that* $k < R(P_i, x_i)$*, there must exists some other* $j \in N$ *with* $R(P_i, x_j) = k$.

The proof is again simple. Suppose not. Then the individual can find an available widget that she likes better, resulting in a Pareto improvement.

An intuitive but useful corollary follows immediately from Lemma 5:

Corollary 1. *For any $x \in PE(P)$ and any $i \in N$, $R(P_i, x_i) \leq N$.*

The next lemma that characterizes a form of upper bounds on the ranking vector of a Pareto efficient allocation.

Lemma 6 (Bounds on Ranking Vectors of Pareto Efficient Allocations)**.** *For any $x \in PE(P)$ and any $k \in \{1, ..., N\}$, there are at least k individuals who are allocated widgets weakly better than their k-th favorite widgets:*

$$\#\{i \in N : \ R(P_i, x_i) \leq k\} \geq k.$$

Corollary 2. *There are at most $(N - k + 1)$ individuals who are allocated widgets weakly worse than their k-th favorite widgets, i.e., $\#\{i \in N : \ R(P_i, x_i) \geq k\} \leq N - k + 1$.*

Based on Lemma 6 and Corollary 2, we can deduce that the ranking vector $R(P, x)$ of any Pareto efficient allocation $x \in PE(P)$ can be bounded above by a permutation of the vector $(1, 2, ..., N)$. It can also be shown that this upper bounded is attainable, and the type of preference profiles that attain this upper bound are $\trianglerighteq$-minimal. See the proofs in Appendices A.5 and A.6 for details.

4.2 Social Preferences

Section 4.1 considers a setting where individuals have strict private preferences over their own widgets but they are completely indifferent over the widgets other individuals obtain. In this section, we seek to incorporate social preferences, where each individual not only cares about the widget she gets herself, but may also care about which widgets other individuals get. However, if we do not impose any structure on social preferences, we are effectively back in a setting with general preferences over a general allocation space, as already considered in Section 3. To investigate partially the middle ground between Section 4.1 and 3, we introduce the following specification.

The set of admissible allocations X is again characterized as the assignment of widgets to individuals, as described in Section 4.1. We assume that each individual i's preference P_i can be factorized into two components $P_i = (P_{i,i}, P_{i,-i})$ where $P_{i,i}$ is a well-defined *strict* preference relation on M while $P_{i,-i}$ is a well-defined preference relation on $X_{-i} := \left\{x_{-i} = (x_j)_{j \neq i} : x \in X\right\}$. Moreover, we require that P_i satisfy the following *lexicographic* structure:

(L1) $x \succ_{P_i} y$ if $x_i \succ_{P_{i,i}} y_i$, or if $x_i = y_i$ and $x_{-i} \succ_{P_{i,-i}} y_{-i}$.

(L2) $x \sim_{P_i} y$ if $x_i = y_i$ and $x_{-i} \sim_{P_{i,-i}} y_{-i}$.

The lexicographic structure of P_i is an extreme modeling device to induce the plausible feature that individuals are primarily concerned with their own widget allocations. The factorization of individual preference and the lexicographic structure jointly allow us to essentially separate the analysis of the *private* component $P_{i,i}$ of each individual i's preference on her own widget from the *social* component $P_{i,-i}$ of her preference on other individuals' widgets, lending great analytical tractability while conveying the key intuition.

Definition 2. Given a preference profile P on X and any individual $i \in N$, we say that $P_{i,-i}$ is "*individualistic*" if, for any $x_{-i}, y_i \in X_{-i}$, we have $x_{-i} \sim_{P_{i,-i}} y_{-i}$.

Clearly, an individual i with an *individualistic* $P_{i,-i}$ is only concerned with her own widget allocation, but is completely indifferent with respect to other individuals' widgets, just as in Section 4.1. In particular, every individual holds no prejudice against another individual's enjoyment of private widgets, even if the definitions of "good widgets" differ dramatically across individuals.

Given any strict private preference profile $(P_{i,i})_{i\in N}$, and any arbitrary social preference profile $(P_{i,-i})_{i\in N}$, we first observe that the set of Pareto efficient allocations under the preference profile $P := (P_{i,i}, P_{i,-i})_{i\in N}$ is invariant with respect to the social preference profile $(P_{i,-i})_{i\in N}$ due to the imposed lexicographic structure.

Formally, define

$$P^{ind} := \left(P_{i,i}, P^{ind}_{i,-i}\right)_{i\in N} \tag{6}$$

and we have the following lemma.

Lemma 7 (Invariance of Pareto Efficient Allocations)**.** $PE\,(P) = PE\left(P^{ind}\right)$.

Lemma (7) allows us to be free of concerns about differences in the sets of Pareto efficient allocations, but focus purely on the ranking vectors. Hence, we may carry over all results from Section 4.1 about the sets of Pareto efficient allocations.

The following theorem shows that P^{ind} is $\trianglerighteq$-maximal among preference profiles that share the same profile of private preferences.

Theorem 3 ($\trianglerighteq$-Maximality of Individualism)**.** *For any* $P = (P_{i,i}, P_{i,-i})_{i\in N}$ *and* $P^{ind} = \left(P_{i,i}, P^{ind}_{i,-i}\right)_{i\in N}$*, we have:*

(a) $P^{ind} \trianglerighteq P$.

(b) $P^{ind} \rhd P$ *if there exist some* $i \in N$ *and two distinct Pareto allocations* $x, y \in PE(P)$ *such that* $x_i = y_i$ *and* $y_{-i} \succ_{P_{i,-i}} x_{-i}$.

(c) $P^{ind} \rhd P$ *if* $N \geq 3$, $\#(PE(P)) \geq 2$ *and* $(P_{i,-i})_{i \in N}$ *is strict.*

The proof of Theorem 3(a) is simple. Consider any Pareto efficient allocation $x \in PE(P)$. Each individual i's ranking of x in X is given by:

$$\begin{aligned} R(P_i, x) &= 1 + \#\{z \in X : z_i \succ_{P_i} x\} \\ &= 1 + \#\left\{z \in X : z_i \succ_{P_{i,i}} x_i\right\} + \#\left\{z \in X : z_i = x_i \text{ and } z_{-i} \succ_{P_{i,-i}} x_{-i}\right\} \\ &\geq 1 + \#\left\{z \in X : z_i \succ_{P_{i,i}} x_i\right\} \\ &= 1 + \#\left\{z \in X : z_i \succ_{P_i^{ind}} x_i\right\} \\ &= R\left(P_i^{ind}, x\right), \end{aligned}$$

where the equality holds if and only if

$$\#\left\{z \in X : z_i = x_i \text{ and } z_{-i} \succ_{P_{i,-i}} x_{-i}\right\} = 0. \tag{7}$$

We may then simply set the mapping $\psi : PE(P) \to PE\left(P^{ind}\right)$ as the identity map and establish that $R\left(P_i^{ind}, \psi(x)\right) \geq R\left(P_i^{ind}, x\right)$ for all x and i.

Clearly, "$\triangleq$" holds if and only if (7) is uniformly true across all $x \in PE\left(P^{ind}\right)$ and all $i \in N$, but (7) is not very explicit and in general involves allocations that may not be Pareto efficient. Intuitively, as the definitions of welfarism and paternalism do not impose any restrictions on $P_{i,-i}$ between any two x_{-i} and y_{-i} that do not Pareto dominate each other, it should be "easy" for (7) to fail and for "$\rhd_X$" to hold.

Theorem 3(b)(c) present two sufficient conditions for "$\rhd_X$" that formalizes this intuition in a more precise manner. In particular, the conditions in Theorem 3(c) are very explicit and arguably weak: $N \geq 3$, $\#(PE(P)) \geq 2$ are almost trivial conditions to check, and the focus on strict $(P_{i,-i})$, though not necessary, covers a representative class of social preferences. The condition in Theorem 3(b) may seem less explicit, but is in fact even weaker than the conditions in Theorem 3(c).

5 A Simple Example with Continuous Allocation Space

We now provide a simple example that illustrates how the key idea underlying the partial order $\trianglerighteq$ proposed in the previous sections for ordinal preference profiles on discrete allocation space can be adapted to settings with cardinal preference profiles on continuous allocation spaces.

Consider again the simple case with two individuals "$1, 2$" and two perfectly divisible goods "$1, 2$". Let $x_i = (x_{i,1}, x_{i,2})$ denotes individual i's consumption of goods $1, 2$, respectively. Suppose now that the society is endowed with 1 unit of each good, so that the feasibility set is given by

$$X = \left\{ (x_1, x_2) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 : \begin{array}{c} x_{1,1} + x_{2,1} \leq 1 \\ x_{1,2} + x_{2,2} \leq 1 \end{array} \right\}.$$

Notice that now the cardinality of X is uncountably infinite, but the dimension is finite. It's now easier to work with cardinal preferences encoded by a *continuous* and *weakly bivariate-increasing* utility function $u_i : \mathbb{R}_+^2 \to \mathbb{R}$.

With continuous u_i and compact X, the levels of maximal and minimal feasible utilities are well-defined:

$$\overline{u}_i = \sup_{x \in X} u_i(x), \quad \underline{u}_i = \inf_{x \in X} u_i(x).$$

Denote the set of feasible utilities as

$$U := u(X).$$
$$U_i := u_i(X) = [\underline{u}_i, \overline{u}_i]$$

Given X and u, we write $PE(u)$ to denote its Pareto frontier. Here:

For each $x \in X$, and any utility function of individual i, we define individual i's ranking evaluation function $r_i(u_i) : X \to [0, 1]$ by

$$r_i(u_i)[x] := \frac{u_i(x) - u_i(x)}{\overline{u}_i - \underline{u}_i}$$

with *"1" corresponding to the most preferred and "0" the least preferred*, as in this example we have

$$-\infty < \underline{u}_i < \overline{u}_i < \infty.$$

We now define a ranking evaluations of PE allocations by

$$rPE_i(u)[x] := \{r_i(u_i)[x] : x \in PE(u)\}.$$

and the Pareto frontier ranking evaluation profiles, $rPE(u, X) : X \to U \subseteq \mathbb{R}^2$, by

$$rPE(u)[x] := \{r(u)[x] : x \in PE(u)\}.$$

For simplicity, let's focus on *private* utilities first for this example.

One "preference profile" $\overline{u}$ is given by

$$\begin{cases} \overline{u}_1(x_1) = x_{1,1} \\ \overline{u}_2(x_2) = x_{2,2} \end{cases}$$

which essentially makes i's and j's preferences "orthogonal" to each other, giving

$$PE(\overline{u}) = \left\{ \left(x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right\}$$

and thus

$$rPE(\overline{u}) = \{(1, 1)\}.$$

An alternative preference profile $\underline{u}$ is given by

$$\underline{u}_i(x_i) = x_{i,1} + x_{i,2}, \ \forall i \in \{1, 2\},$$

which essentially makes i, j's utility function "coincide perfectly", giving

$$PE(\underline{u}) = \left\{ \left(x_1 = \begin{pmatrix} s \\ t \end{pmatrix}, x_2 = \begin{pmatrix} 1-s \\ 1-t \end{pmatrix} \right) : s, t \in [0, 1] \right\}$$
$$U_i = [0, 2]$$

and

$$\begin{aligned} rPE(\underline{u})[x] &= \left\{ \left(\frac{\underline{u}_1(x) - 0}{2 - 0}, \frac{\underline{u}_2(x) - 0}{2 - 0}, \right) : x \in PE(\underline{u}), x_1 = \begin{pmatrix} s \\ t \end{pmatrix} \right\}. \\ &= \left\{ \left(\frac{s+t}{2}, 1 - \frac{s+t}{2}, \right) : s, t \in [0, 1] \right\} \\ &= \{(t, 1-t) : t \in [0, 1]\} \end{aligned}$$

The two preference profiles $\overline{u}$ and $\underline{u}$ can then be "ordered" based on the observation that

$$rPE(\overline{u}) \geq rPE(\underline{u})[x], \quad \forall x \in PE(\underline{u}).$$

6 Conclusion

This paper proposes a theoretical framework under which preference profiles can be meaningfully compared, without the need to aggregate or trade off preferences across individuals. The current set of models and results above are intended more as illustrations of the general idea, and there are clearly many directions for further explorations.

First, it would be interesting to generalize the illustrative example in Section (5) so as to accommodate canonical continuous allocation spaces and profiles of continuous utility functions, and investigate how different configurations of social preferences can be evaluated under the partial order. In particular, it would be theoretically appealing if we can incorporate a "numeraire", on which every individual's preference is perfectly aligned (e.g. everyone prefers more money than less). Additionally, the framework can be further enriched by introducing endowments into the setup, which helps establish a selection from or a refinement of the Pareto frontier.

Second, it would be interesting to analyze various forms of social preferences, potentially in a cardinal framework without the lexicographic structure imposed between private preferences and social preferences in Section 4.2. For example, a *paternalistic* social preference would be one such that an individual i is happier when other individuals obtain private allocations that are more favorable according to i's preference. A *welfaristic* social preference would be one such that an individual i is happier when other individuals obtain allocations that are more favorable according to other individuals' own preferences.

Lastly, the current framework focuses on a fixed set of individuals. Alternatively, one can ask whether the partial order can be adapted to a setting with a large number (or a distribution) of individuals with a corresponding distribution of preferences (or "types"). If the word "ideology" can be interpreted as a distribution over preferences, at least in certain contexts, then the framework may potentially serve as a formal basis for the comparison of different ideologies.

References

Belmonte, L. A. (2020): *The International LGBT Rights Movement: A History*, Bloomsbury Publishing.

Binmore, K. (1989): "Social Contract I: Harsanyi and Rawls," *The Economic Journal*, 99, 84–102.

Camerer, C. (2008): "The case for mindful economics," *Foundations of positive and normative economics*, 43–69.

Caplin, A. and A. Schotter (2008): *The foundations of positive and normative economics: a handbook*, Oxford University Press.

Gul, F. and W. Pesendorfer (2008): "The case for mindless economics," *The foundations of positive and normative economics: A handbook*, 1, 3–42.

Harsanyi, J. C. (1975): "Can the maximin principle serve as a basis for morality? A critique of John Rawls's theory," *American political science review*, 69, 594–606.

Hausman, D. (2008): "Mindless or mindful economics: A methodological evaluation," *The foundations of positive and normative economics: A handbook*, 125–55.

Milgrom, P. and C. Shannon (1994): "Monotone comparative statics," *Econometrica: Journal of the Econometric Society*, 157–180.

Postlewaite, A. (2011): "Social Norms and Preferences," in *Handbook of Social Economics*, ed. by J. Benhabib, A. Bisin, and M. Jackson, Elsevier, vol. 1, chap. 2, 31–67.

Rawls, J. (1958): "Justice as fairness," *The philosophical review*, 67, 164–194.

Topkis, D. M. (1998): *Supermodularity and Complementarity*, Princeton University Press.

A Main Proofs

A.1 Proof of Lemma 1

Proof. Suppose that $X \backslash PE(P,X) \neq \emptyset$. Take any Pareto dominated allocation $\underline{x} \in X \backslash PE(P,X)$, and define

$$\begin{aligned}
\overline{PE}_{\underline{x}}(P) &:= \{y \in PE(P) : R(P,y) \lneqq R(P,\underline{x})\}, \\
\underline{PE}_{\underline{x}}(P) &:= PE(P) \backslash \overline{PE}_{\underline{x}}(P), \\
NPE(P) &:= X \backslash PE(P).
\end{aligned}$$

Clearly, $\overline{PE}_{\underline{x}}(P, X) \neq \emptyset$ and

$$X = \overline{PE}_{\underline{x}}(P) \cup \underline{PE}_{\underline{x}}(P) \cup NPE(P).$$

Take any $x^* \in \overline{PE}_{\underline{x}}(P, X)$. there must exist some individual $\bar{i} \in N$ such that

$$R(P_{\bar{i}}, x^*) < R(P_{\bar{i}}, \underline{x}).$$

Now, define

$$\overline{x} :\in \arg \min_{y \in \overline{PE}_{\underline{x}}(P)} R(P_{\bar{i}}, y)$$

Define

$$ID_{\overline{x}} := \{x \in X : x \sim_{P_i} \overline{x} \text{ for all } i\}$$

We must have

$$R(P, \overline{x}) \leq R(P, \underline{x}),$$

and moreover

$$R(P_{\bar{i}}, \overline{x}) \leq R(P_{\bar{i}}, x^*) < R(P_{\bar{i}}, \underline{x}).$$

We now construct another preference profile P' by switching individual $\bar{i}$'s rankings of $\underline{x}$ and $\overline{x}$ only and keep all other rankings unchanged:

$$\begin{aligned} &R\left(P'_{\bar{i}}, \underline{x}\right) := R(P_{\bar{i}}, \overline{x}), \\ &R\left(P'_{\bar{i}}, \overline{x}\right) := R(P_{\bar{i}}, \underline{x}) - \#ID_{\overline{x}}(+1), \\ &R\left(P'_{\bar{i}}, x\right) := R(P_{\bar{i}}, x) - \#ID_{\overline{x}} \text{ for } x \text{ s.t. } \overline{x} \succ_{P_{\bar{i}}} x \succ_{P_{\bar{i}}} \underline{x} \qquad (8) \\ &R\left(P'_{\bar{i}}, x\right) := R(P_{\bar{i}}, x), \ \forall x \in X \setminus \{\underline{x}, \overline{x}\}, \\ &R\left(P'_{i}, x\right) := R(P_{i}, x), \ \forall i \in N \setminus \left\{\bar{i}\right\}, \forall x \in X. \qquad (9) \end{aligned}$$

Notice that this construction ensures that

$$R\left(P', \underline{x}\right) \lneqq R(P, \underline{x}), \qquad (10)$$

$$R\left(P', \overline{x}\right) \gneqq R(P, \overline{x}), \qquad (11)$$

$$R\left(P', \underline{x}\right) \geq R(P, \overline{x}). \qquad (12)$$

We now show that $\underline{x} \in PE\left(P'\right)$. First, notice that $\underline{x}$ is not Pareto dominated by any allocation $x \in \overline{PE}_{\underline{x}}(P)$ under P': by construction,

$$\begin{aligned} R\left(P_{\bar{i}}', \underline{x}\right) &= R\left(P_{\bar{i}}, \overline{x}\right) \\ &\leq R\left(P_{\bar{i}}, x\right), \ \forall x \in \overline{PE}_{\underline{x}}(P) \setminus \{\overline{x}\}, \\ &= R\left(P_{\bar{i}}', x\right), \ \forall x \in \overline{PE}_{\underline{x}}(P) \setminus \{\overline{x}\}. \end{aligned}$$

Moreover, $\underline{x}$ is not dominated by any allocation $x \in \left(X \backslash \overline{PE}_{\underline{x}}(P)\right) \setminus \{\underline{x}\}$ under P': if $x \in X \backslash \overline{PE}_{\underline{x}}(P)$ and $x \neq \underline{x}$, then either (i) there must exist some $i \in N$ such that $R(P_i, x) > R(P_i, \underline{x})$,or (ii): $R(P, x) = R(P, \underline{x})$. For case (i), by (10) we must have

$$R\left(P_i', x\right) = R(P_i, x) > R(P_i, \underline{x}) \geq R\left(P_i', \underline{x}\right)$$

For case (ii), by (10) we must have

$$R\left(P', x\right) = R(P, x) = R(P, \underline{x}) \geq R(P, \underline{x}).$$

In either case, x cannot Pareto dominate $\underline{x}$.

Next we show that $PE(P) \setminus \{\overline{x}\} \subseteq PE\left(P'\right)$. For any $x \in PE(P) \setminus \{\overline{x}\}$, it is not Pareto dominated by any $x' \in X \setminus \{\underline{x}\}$ under P, so it must not be Pareto dominated by x under P', as the ranking profile of x stay unchanged while the ranking profile of x' weakly worsens:

$$R\left(P', x'\right) \geq R\left(P, x'\right), \ \forall x' \in X \setminus \{\underline{x}\}.$$

In particular, recall that by (11) the ranking profile of $\overline{x}$ strictly worsens from P to P'. Moreover, x cannot be Pareto dominated by $\underline{x}$ either. We consider three cases separately.

Case (i): $x \in \underline{PE}_{\underline{x}}(P)$, i.e., x is Pareto efficient but x does not Pareto dominate $\underline{x}$. In this case,

$$R(P, x) \neq R(P, \underline{x}), \tag{13}$$

otherwise x would be Pareto dominated by $\overline{x}$ under P. Then, given $\underline{x}$ is not Pareto dominated by x under P and (13), there must exist some $i \in N$ such that

$$R(P_i, x) > R(P_i, \underline{x}) \geq R(P_i, \overline{x}).$$

Now, given that x is not Pareto dominated by $\overline{x}$ under P, there must exist some $j \in N$ such that

$$R(P_j, x) < R(P_j, \overline{x}).$$

As $x \notin \{\underline{x}, \overline{x}\}$, $R\left(P', x\right) = R(P, x)$ by (9), we have, if $j = \overline{i}$,

$$R\left(P_j', x\right) < R\left(P_j, \overline{x}\right) = R\left(P_j', \underline{x}\right),$$

and if $j \neq \overline{i}$,

$$R\left(P_j', x\right) < R\left(P_j, \overline{x}\right) \leq R\left(P_j, \underline{x}\right) \leq R\left(P_j', \underline{x}\right),$$

implying that x cannot be Pareto dominated by $\underline{x}$ under P'.

Now, for $x \in \overline{PE}_{\underline{x}}(P) \setminus \{\overline{x}\}$, as x and $\overline{x}$ do not Pareto dominate each other, we have two additional cases.

Case (ii): $x \in \overline{PE}_{\underline{x}}(P) \setminus \{\overline{x}\}$ and there exists some $j \in N$ such that

$$R\left(P_j, x\right) < R\left(P_j, \overline{x}\right).$$

Then we can apply the same arguments as in Case (i) and deduce that

$$R\left(P_j', x\right) < R\left(P_j', \underline{x}\right),$$

implying that x cannot be Pareto dominated by $\underline{x}$ under P'.

Case (iii): $x \in \overline{PE}_{\underline{x}}(P) \setminus \{\overline{x}\}$ and $R(P, x) = R(P, \overline{x})$, i.e., all individuals are indifferent between x and $\overline{x}$. Then, as $x \notin \{\overline{x}, \underline{x}\}$,

$$\begin{aligned} R\left(P', x\right) &= R(P, x) = R(P, \overline{x}) \\ &\begin{cases} = R\left(P_{\overline{i}}', \underline{x}\right), \\ \geq R\left(P_j, \underline{x}\right), \quad j \neq \overline{i} \end{cases} \\ &\geq R\left(P', \underline{x}\right), \end{aligned}$$

implying that x cannot be Pareto dominated by $\underline{x}$ under P'.

In summary of the above, we have shown that

$$PE\left(\underline{P}\right) \cup \{\underline{x}\} \setminus \{\overline{x}\} \subseteq PE\left(P'\right)$$

which in particular implies that $\#\left(PE\left(\underline{P}\right)\right) \leq \#\left(PE\left(P'\right)\right)$.

Now consider the set

$$\Delta := PE\left(P'\right) \setminus \left(PE(P) \cup \{\underline{x}\}\right),$$

which may or may not be empty. In particular, notice that $\overline{x} \notin \Delta$ as $\overline{x} \in PE(P)$.

Suppose that $\Delta \neq \emptyset$. Take any $x \in \Delta$. Clearly, $x \in NPE(P)$. We show that x must be Pareto dominated by $\overline{x}$ under P, i.e.,

$$R(P, x) \gneqq R(P, \overline{x}).$$

To see this, notice that the ranking profile of x must have remained unchanged from P to $P^{'}$, while only the ranking profile of $\overline{x}$ in X has worsened from P to $P^{'}$, i.e.

$$\begin{aligned}
&R\left(P^{'}, x\right) = R(P, x), \\
&R\left(P^{'}, \overline{x}\right) \gneqq R(P, \overline{x}), \\
&R\left(P^{'}, y\right) \leq R(P, y), \ \forall y \in X \backslash \{\overline{x}\}.
\end{aligned}$$

As a result, if x is Pareto dominated by some $y \in X \backslash \overline{x}$ under P, x must still be Pareto dominated by y under $P^{'}$. Hence, by the fact that x is Pareto dominated under P but becomes Pareto efficient under $P^{'}$, it must be the case that x is Pareto dominated by $\overline{x}$ under P. Then, we must have

$$R\left(P^{'}, x\right) = R(P, x) \gneqq R(P, \overline{x}). \tag{14}$$

Now, observe that

$$PE\left(P^{'}\right) = [PE(P) \setminus \{\overline{x}\}] \cup \{\underline{x}\} \cup \Delta \cup \left[PE\left(P^{'}\right) \cap \{\overline{x}\}\right],$$

where the last term $PE\left(P^{'}\right) \cap \{\overline{x}\}$ may either be nonempty or empty, depending on whether $\overline{x} \in PE\left(P^{'}\right)$.

We now define the mapping $\psi : PE\left(P^{'}, X\right) \to PE(P, X)$ by

$$\begin{aligned}
\psi(x) &:= \begin{cases} x, & x \in PE(P) \setminus \{\overline{x}\}, \\ \overline{x}, & x \in \{\underline{x}\}, \\ \overline{x}, & x \in \Delta, \\ \overline{x}, & x \in PE\left(P^{'}\right) \cap \{\overline{x}\}, \end{cases} \\
&= \begin{cases} x, & x \in PE(P) \setminus \{\overline{x}\}, \\ \overline{x}, & x \in PE\left(P^{'}\right) \setminus (PE(P) \setminus \{\overline{x}\}), \end{cases}
\end{aligned}$$

which is clearly an onto mapping:

$$\psi\left(PE\left(P^{'}\right)\right) = PE\left(P\right).$$

Moreover, for every $x \in PE\left(P^{'}\right)$, we have

$$\begin{aligned}
R\left(P,\psi\left(x\right)\right) &= \begin{cases} R\left(P,x\right), & x \in PE\left(P\right)\setminus\left\{\overline{x}\right\}, \\ R\left(P,\overline{x}\right) & x \in \left\{\underline{x}\right\}, \\ R\left(P,\overline{x}\right) & x \in \Delta, \\ R\left(P,\overline{x}\right) & x \in PE\left(P^{'}\right)\cap\left\{\overline{x}\right\}, \end{cases} \\
&\sim \begin{cases} = R\left(P^{'},x\right), & x \in PE\left(P\right)\setminus\left\{\overline{x}\right\}, \quad \text{by (9)}, \\ \leq R\left(P^{'},\underline{x}\right), & x \in \left\{\underline{x}\right\}, \quad \text{by (12)}, \\ < R\left(P^{'},x\right), & x \in \Delta, \quad \text{by (14)}, \\ \lneqq R\left(P^{'},\overline{x}\right) & x \in PE\left(P^{'}\right)\cap\left\{\overline{x}\right\}, \quad \text{by (11)}, \end{cases} \\
&= R\left(P^{'},x\right).
\end{aligned}$$

In summary, we have established the existence of an onto mapping $\psi : PE\left(P^{'}\right) \to PE\left(P\right)$ such that

$$R\left(P,\psi\left(x\right)\right) \leq R\left(P^{'},x\right),\ \forall x \in PE\left(P^{'}\right)$$

with at least one strict inequality. Hence, $P \succ_X P^{'}$, so P is not a minimal element. □

A.2 Proof of Lemma 2

Proof. Suppose that there exist two distinct individuals $i, j \in N$ and a Pareto efficient allocation $x \in PE\left(P\right)$ such that

$$x \sim_{P_i} y,\ x \sim_{P_j} y$$

for some $y \in X\setminus\left\{x\right\}$. We construct another preference profile $P^{'}$ in the following way.

Construction of $P^{'}$

We keep the preferences of all individuals other than i, j unchanged, i.e., $P_k^{'} = P_k$ for all $k \in N\setminus\left\{i,j\right\}$.

For individual i, we construct $P_i^{'}$ by perturbing P_i in the following way. Define

$$\mathcal{I}_x\left(P_i\right) := \left\{z \in X\setminus\left\{x\right\} : z \sim_{P_i} x\right\},$$

First, we set

$$x \succ_{P_i'} y.$$

Second, if there exists any $z \in \mathcal{I}_x(P_i) \setminus \{y\}$, we set

$$z \sim_{P_i'} y.$$

In other words, the allocation x is preferred under $P_i^{'}$ to any other allocation z that i found indifferent with x under P_i, i.e.,

$$x \succ_{P_i'} z, \quad \forall z \in \mathcal{I}_x(P_i).$$

Third, we keep all other pairwise preferences in P_i unchanged.

For individual j, we construct $P_j^{'}$ by perturbing P_j in the following way. Now, write

$$\mathcal{I}_x(P_j) := \left\{ z \in X \setminus \{x\} : z \sim_{P_j} x \right\},$$

First, we set

$$y \succ_{P_j'} x.$$

Second, if there exists any $z \in \mathcal{I}_x(P_j) \setminus \{y\}$, we set

$$z \sim_{P_j'} y.$$

In other words, any other allocation z that j found indifferent with x under P_j, is now preferred under $P_j^{'}$ to allocation x:

$$z \succ_{P_j'} x, \quad \forall z \in \mathcal{I}_x(P_j).$$

Note that, for any two allocations $u, v \in X$, individual preferences between u and v under P and $P^{'}$ can be different only if $u = x$ and $v \in \mathcal{I}_x(P_i) \cup \mathcal{I}_x(P_j)$ or vice versa.

Characterization of $PE\left(P^{'}\right)$

We claim that

$$PE\left(P^{'}\right) = PE(P) \cup \{z \in \mathcal{I}_x(P_j)\text{: } z \text{ is Pareto dominated under } P \text{ by } x \text{ and only } x\}.$$

We prove this claim by considering the following cases:

1. We show

$$x \in PE\left(P^{'}\right).$$

Recall that $x \in PE(P)$ by supposition, and consider the following cases:

(a) For any $z \in \mathcal{I}_x(P_i)$, i.e., $z \sim_{P_i} x$, we know that $x \succ_{P_i'} z$, so x cannot be Pareto dominated by z.

(b) For any $z \in \mathcal{I}_x(P_j) \setminus \mathcal{I}_x(P_i)$, i.e., $z \sim_{P_j} x$ but $z \nsim_{P_i} x$,, we consider the following two possibilities. If $x \succ_{P_i} z$, then this pairwise comparison remain unchanged from P to $P^{'}$, so we have $x \succ_{P_i'} z$ by the construction of $P^{'}$ and thus x cannot be Pareto dominated by z. If otherwise $z \succ_{P_i} x$, then we can deduce from $x \in PE(P)$ that there must exist some individual $k \notin \{i,j\}$ with $x \succ_{P_k} z$. Again, since we did not perturb the preference of any $k \notin \{i,j\}$in the construction of $P^{'}$, we have $x \succ_{P_k'} z$, so x cannot be Pareto dominated by z.

(c) For any $z \notin \mathcal{I}_x(P_i) \cup \mathcal{I}_x(P_j) \cup \{x\}$, i.e. $z \nsim_{P_i} x$ and $z \nsim_{P_j} x$, the pairwise comparison between x and z stays completely unchanged from P to $P^{'}$. Hence, given that x was not Pareto dominated by z under P, it remains under $P^{'}$.

Combining the three collectively exhaustive cases above, we conclude that x is not Pareto dominated by any $x \in X \setminus \{x\}$ under $P^{'}$, or equivalently, $x \in PE\left(P^{'}\right)$.

2. We show

$$PE(P) \cap \mathcal{I}_x(P_i) \cap \mathcal{I}_x(P_j) \subseteq PE\left(P^{'}\right).$$

To see this, consider any $z \in PE(P) \cap \mathcal{I}_x(P_i) \cap \mathcal{I}_x(P_j)$.

(a) For x, by the construction of P' we have $x \succ_{P_i'} z$ and $z \succ_{P_j'} x$, so z cannot be dominated by x under $P^{'}$.

(b) For any $w \in \mathcal{I}_x(P_i) \cap \mathcal{I}_x(P_j)$, we have $z \sim_{P_i'} w$ and $z \sim_{P_j'} w$ by the construction of $P^{'}$ and the preferences between z and w of any other individuals $k \notin \{i,j\}$ stay unchanged from P to $P^{'}$. Hence, if z is not Pareto dominated by x under P, it remains so under $P^{'}$.

(c) For any $w \in \mathcal{I}_x(P_i) \setminus \mathcal{I}_x(P_j)$, we consider two possibilities given $w \notin \mathcal{I}_x(P_j)$. If $z \succ_{P_j} w$, then $z \succ_{P_j'} w$ and thus z is not Pareto dominated by w under $P^{'}$. Otherwise if $w \succ_{P_j} z$, then by $z \in PE(P) \cap \mathcal{I}_x(P_i)$ there must exist some $k \notin \{i,j\}$ such that $z \succ_{P_k} w$. Then $P_k^{'} = P_k$ by construction and $z \succ_{P_k'} x$, so again z is not Pareto dominated by w under $P^{'}$.

(d) For any $w \in \mathcal{I}_x(P_j) \setminus \mathcal{I}_x(P_i)$, the arguments above in 2(c) apply with i in place of j.

(e) For any $w \in X \backslash (\{x\} \cup \mathcal{I}_x(P_i) \cup \mathcal{I}_x(P_j))$, the preference between z and w of any individual remains unchanged. Hence, if z is not Pareto dominated by x under P, it remains so under $P^{'}$.

Combing the five collectively exhaustive cases above, we conclude that $z \in PE\left(P^{'}\right)$.

3. We show

$$PE(P) \cap \mathcal{I}_x(P_i) \backslash \mathcal{I}_x(P_j) \subseteq PE\left(P^{'}\right).$$

To see this, consider any $z \in PE(P) \cap \mathcal{I}_x(P_i) \backslash \mathcal{I}_x(P_j)$.

(a) For x, we consider two possibilities given that $z \notin \mathcal{I}_x(P_j)$. If $z \succ_{P_j} x$, then we have $z \succ_{P_j^{'}} x$ and thus z is not Pareto dominated by x under $P^{'}$. Otherwise if $x \succ_{P_j} z$, then since $z \in PE(P) \cap \mathcal{I}_x(P_i)$ there must exist some $k \notin \{i, j\}$ such that $z \succ_{P_k} x$. Then $P_k^{'} = P_k$ by construction and $z \succ_{P_k^{'}} x$, so again z is not Pareto dominated by x under $P^{'}$.

(b) For $w \in X \backslash \{x\}$, no individual's preference between z and w has been changed from P to $P^{'}$. (In particular, if $w \sim_{P_i} z \sim_{P_i} x$, then under $P^{'}$ we have $x \succ_{P_i^{'}} w \sim_{P_i^{'}} z$, so the preference between z and w stays unchanged from P to $P^{'}$.[3]

4. Similarly to case 3, we have

$$PE(P) \cap \mathcal{I}_x(P_j) \backslash \mathcal{I}_x(P_i) \subseteq PE\left(P^{'}\right).$$

5. We have

$$PE(P) \backslash (\{x\} \cup \mathcal{I}_x(P_i) \cup \mathcal{I}_x(P_j)) \subseteq PE\left(P^{'}\right),$$

since every individual's preference between $z \in PE(P) \backslash (\{x\} \cup \mathcal{I}_x(P_i) \cup \mathcal{I}_x(P_j))$ and any $w \in X$ stays unchanged from P to $P^{'}$.\

[3]This can be seen more clearly by considering the following three sub-cases separately:

i. For $w \in \mathcal{I}_x(P_i) \backslash \mathcal{I}_x(P_j)$, we have $z \sim_{P_i^{'}} w$ and no other individual's preference between z and w is changed from P to $P^{'}$. Hence, given that z is not Pareto dominated by w under P, it remains so under $P^{'}$.

ii. For $w \in \mathcal{I}_x(P_j) \backslash \mathcal{I}_x(P_i)$, we know that both i's and j's preferences betweeen z and w stay unchanged from P to $P^{'}$. Hence, given that z is not Pareto dominated by w under P, it remains so under $P^{'}$.

iii. For any $w \in X \backslash (\{x\} \cup \mathcal{I}_x(P_i) \cup \mathcal{I}_x(P_j))$, we note that the comparison between w and x does not change at all from P to $P^{'}$. Hence, given that z was not Pareto dominated by w under P, it remains so under $P^{'}$.

Combining Points 1-5 above, we have

$$PE(P) \subseteq PE\left(P'\right). \tag{15}$$

Now we analyze what happens to every $z \notin PE(P)$ after the change from P to P'. Specifically, we separately consider the following possibilities:

1. If $z \in X \backslash (PE(P) \cup \mathcal{I}_x(P_j))$, we can show that

$$z \notin PE\left(P'\right).$$

   To see this, notice that $z \notin PE(P)$ implies that z is Pareto dominated by some allocation $w \in X$. If $w \neq x$, then z will remain Pareto dominated by w, since everyone's preference between z and w stays unchanged from P to P'. If $w = x$, then from P to P' an individual's preference of z relative to x can strictly improve only for individual j and only if $z \in \mathcal{I}_x(P_j)$; however, since $z \notin \mathcal{I}_x(P_j)$, we conclude that z remains Pareto dominated by x under P'.

2. If $z \in (X \backslash PE(P)) \cap \mathcal{I}_x(P_j)$ and z is Pareto dominated by some $w \in X \backslash \{x\}$ under P, we can again show that

$$z \notin PE\left(P'\right).$$

   To see this, notice that every individual's preference between z and w stays unchanged from P to P'. Hence, given that z is Pareto dominated by w under P, it remains so under P'.

3. If $z \in (X \backslash PE(P)) \cap \mathcal{I}_x(P_j)$ and z is Pareto dominated by x and only x under P, we now show that

$$z \in PE\left(P'\right).$$

   To see this, notice that z cannot be Pareto dominated by x under P', since we have set

$$z \succ_{P'_j} x$$

   given that $z \in \mathcal{I}_x(P_j)$. In the meanwhile, since z is not Pareto dominated by any $w \in X \backslash \{x\}$ under P, it remains so under P', given that every individual's preference between z and w stays unchanged from P to P'.

Combining (15) with Points 6-8 above, we deduce that

$$PE\left(P'\right) = PE(P) \cup \{z \in \mathcal{I}_x(P_j) \colon z \text{ is Pareto dominated under } P \text{ by } x \text{ and only } x\}.$$

Construction of Mapping ϕ and Proof of Ordering

We now construct the mapping $\phi : PE\left(P'\right) \to PE(P)$ by

$$\phi(z) = \begin{cases} z, & z \in PE(P), \\ x, & z \in PE\left(P'\right) \backslash PE(P). \end{cases}$$

and prove that

$$R\left(P', z\right) \geq R(P, \phi(z)) \quad \forall z \in PE\left(P'\right),$$

with at least one strictly inequality:

1. For $z = x$, we have $\phi(x) = x$ and

$$\begin{aligned} R\left(P_i', x\right) &= R(P_i, x), \\ R\left(P_j', x\right) &= R(P_j, x) + \#\left(\mathcal{I}_x(P_j)\right) \\ &\geq R(P_j, x) + 1 > R(P_j, x), \\ R\left(P_k', x\right) &= R(P_k, x), \quad \forall k \in N \backslash \{i, j\}. \end{aligned}$$

   since we know $y \in \mathcal{I}_x(P_j)$ and thus $\#\left(\mathcal{I}_x(P_j)\right) \geq 1$ and thus the inequality $R\left(P_j', x\right) > R(P_j, x)$ must be strict.

2. For $z \in PE(P) \cap \mathcal{I}_x(P_i)$, we have $\phi(z) = z$ and

$$\begin{aligned} R\left(P_i', z\right) &= R\left(P_i', x\right) + 1 \\ &= R(P_i, x) + 1 > R(P_i, x) = R(P_i, z), \\ R\left(P_j', z\right) &= R(P_j, z), \\ R\left(P_k', x\right) &= R(P_k, x), \quad \forall k \in N \backslash \{i, j\}. \end{aligned}$$

3. For $z \in PE(P) \backslash \left[\{x\} \cup \mathcal{I}_x(P_i)\right]$, we have $\phi(z) = z$ and

$$\begin{aligned} R\left(P_i', z\right) &= R(P_i, z), \\ R\left(P_j', z\right) &= R(P_j, z), \\ R\left(P_k', x\right) &= R(P_k, x), \quad \forall k \in N \backslash \{i, j\}. \end{aligned}$$

4. For $z \in PE\left(P'\right) \backslash PE(P)$, we know that $z \in \mathcal{I}_x(P_j)$ and z is Pareto dominated by x

and only x under P. Hence, recalling that $\phi(z) = x$, we have

$$\begin{aligned}
R\left(P_i^{'}, z\right) &\geq R\left(P_i, z\right) \geq R\left(P_i, x\right), \\
R\left(P_j^{'}, z\right) &= R\left(P_j, z\right) \geq R\left(P_j, x\right), \\
R\left(P_k^{'}, z\right) &= R\left(P_k, z\right) \geq R\left(P_k, x\right), \quad \forall k \in N \backslash \{i, j\}.
\end{aligned}$$

Hence, P is not minimal. □

A.3 Proof of Lemma 3

Proof. Suppose that there exists a Pareto efficient allocation $x \in PE(P)$ such that $x \sim_{P_i} y$ for some i and some $y \neq x$.

If there exists another individual $j \neq i$ such that $x \sim_{P_j} y$, then the condition of Lemma 2 is satisfied, so the proof and conclusion of Lemma 2 apply.

Hence, here we focus on the remaining case, in which

$$x \nsim_{P_j} y, \quad \forall j \neq i. \tag{16}$$

We construct another preference profile $P^{'}$ by setting

$$x \succ_{P_i^{'}} y \sim_{P_i^{'}} z, \quad \forall z \in \mathcal{I}_x(P_i) \backslash \{y\},$$

where

$$\mathcal{I}_x(P_i) := \{z \in X \backslash \{x\} : z \sim_{P_i} x\}.$$

We keep all other pairwise preference relations completely unchanged from P to $P^{'}$.

We claim that $PE\left(P^{'}\right) = PE(P)$.

We first show that $PE(P) \subseteq PE\left(P^{'}\right)$ by considering the following three cases separately:

1. First, x is clearly not Pareto dominated under $P^{'}$ by construction.

2. Second, we show that any $z \in PE(P) \cap \mathcal{I}_x(P_i)$ is not Pareto dominated under $P^{'}$. To see this, notice first that z is not Pareto dominated by x under P. By (16), there is no other individual j such that $z \sim_{P_j} x$. Hence, there must exist two individuals $j, k \in N \backslash \{i\}$ such that
$$x \succ_{P_j} z, \quad \text{and} \quad z \succ_{P_k} x.$$

Since j,k's preferences stay unchanged from P to $P^{'}$,it follows that

$$x \succ_{P_j^{'}} z, \quad \text{and} \quad z \succ_{P_k^{'}} x,$$

implying that z is not Pareto dominated by x. As the preference relation between z and any other $w \in X\backslash \{x,z\}$ is unchanged from P to $P^{'}$, we conclude that $z \in PE\left(P^{'},X\right)$.

3. Third, any $z \in X\backslash [\{x\} \cup \mathcal{I}_x(P_i)]$ cannot be Pareto dominated under $P^{'}$, as the preferences between z and any other $w \in X\backslash \{z\}$ stays unchanged from P to $P^{'}$.

Combining Points 1-3, we deduce that $PE(P) \subseteq PE\left(P^{'}\right)$.

We now show that $X\backslash PE(P) \subseteq X\backslash PE\left(P^{'}\right)$.

For any $z \in X\backslash PE(P)$, it must be Pareto dominated by some $w \in PE(P)$. If $w = x$, then z remains Pareto dominated by x under $P^{'}$, since from P to $P^{'}$ the ranking vector of x has weakly improved while the ranking vector of z cannot increase. If $w \neq x$, then every individual's preference between z and w stays unchanged from P to $P^{'}$, so z must remain Pareto dominated by w under $P^{'}$.

Hence, we conclude that $PE\left(P^{'}\right) = PE(P)$.

Setting $\psi : PE\left(P^{'},X\right) \to PE(P,X)$ as the identity mapping $\psi(z) = z$ for all $z \in PE(P,X)$, we have

$$\begin{aligned}
R\left(P_i^{'},x\right) &= R(P_i,x), \\
R\left(P_i^{'},z\right) &= R(P_i,z) + 1 > R(P_i,z), \quad \forall z \in \mathcal{I}_x(P_i), \\
R\left(P_i^{'},z\right) &= R(P_i,z), \quad \forall z \in X\backslash (\{x\} \cup \mathcal{I}_x(P_i)), \\
R\left(P_j^{'},z\right) &= R(P_j,z) \quad \forall j \neq i, \forall z \in X,
\end{aligned}$$

and thus $R\left(P^{'},\psi(z)\right) \geq R(P,z)$ for all $z \in PE(P,X)$ with at least one strict inequality. □

A.4 Proof of Theorem 1

Proof.

(a) $R\left(\overline{P},x^*\right) = \mathbf{1}_N \leq R(P,x)$ for all $x \in X$ and all possible preference profile P.

(b) The "only if" part immediate from Lemma 1 and Lemma 3. Here we prove the "if" part:

Given any other preference profile $P^{'}$, we show that it cannot be the case that $P \succ_X P^{'}$. We prove by contradiction and suppose that $P \succ_X P^{'}$. First, notice that by the supposition that $PE(\underline{P}) = X$, for there to exist an onto mapping $\psi : PE\left(P^{'}\right) \rightarrow PE(\underline{P})$, it must be the case that

$$PE\left(P^{'}\right) = PE(\underline{P}) = X,$$

and ψ is a permutation mapping on X. Moreover, by the supposition we must have

$$R\left(P^{'}, x\right) \geq R(\underline{P}, \psi(x)), \quad \text{for all } x \in X.$$

and

$$R\left(P^{'}, x\right) \gneqq R(\underline{P}, \psi(x)), \quad \text{for some } x \in X.$$

Then, by the summing over all individuals $i \in N$ and all $x \in X$, we have

$$\sum_{i \in N} \sum_{x \in X} R\left(P_i^{'}, x\right) > \sum_{i \in N} \sum_{x \in X} R(\underline{P}_i, \psi(x)) \tag{17}$$

By the supposition that $\underline{P}$ is strict, each individual's ranking of $x \in X$ must be a permutation of $(1, ..., M)$, while under a general $P^{'}$, each ranking vector must be weakly dominated by $(1, ..., M)$, so we have

$$\sum_{i \in N} \sum_{x \in X} R\left(P_i^{'}, x\right) \leq \frac{1}{2} NM(M+1) = \sum_{i \in N} \sum_{x \in X} R(\underline{P}_i, \psi(x)),$$

contradicting (17). Hence, $\underline{P}$ must be minimal.

□

A.5 Proof of Lemma 6

Proof. Suppose not. Write

$$L := \{i \in N : \ R(P_i, x_i) \leq k\},$$
$$H := \{i \in N : R(P_i, x_i) \geq k+1\}.$$

Then there is some $k \in \{1, ..., N\}$ such that $\#(L) < k$. As $\#(L) + \#(H) \equiv N$, then $\#(H) \geq N - k + 1$. Take any individual $h_1 \in H$. and write

$$Q_{h_1} := \left\{z \in M : z \succ_{P_{h_1}} x_{h_1}\right\},$$

to denote the set of widgets that individual h_1 ranks higher than x_{h_1}. By Lemma 5, all widgets in Q_{h_1} must have been assigned to someone in

$$N \backslash \{h_1\} = L \cup (H \backslash \{h_1\}).$$

By construction, $\#(L) < k$ but

$$\#(Q_{h_1}) = R(P_{h_1}, x_{h_1}) - 1 \geq k,$$

so there is at least one individual $h_2 \in H \backslash \{h_1\}$ with $x_{h_2} \in Q_{h_1}$,i.e.,

$$x_{h_2} \succ_{P_{h_1}} x_{h_1}.$$

Now, by Lemma 4, h_2 must also like her own widget x_{h_2} better than h_1's widget x_{h_1},

$$x_{h_2} \succ_{P_{h_2}} x_{h_1}., \tag{18}$$

otherwise it would be a Pareto improvement for h_1 and h_2 to exchange their widgets.

Consider

$$Q_{h_2} := \left\{ z \in M : z \succ_{P_{h_2}} x_{h_2} \right\}.$$

with $\#(Q_{h_2}) \geq k$ again. Again, all widgets in Q_{h_2} must have been assigned to someone else. However, by (18), $x_{h_1} \notin Q_{h_2}$. Hence, all widgets in Q_{h_2} must have been assigned to some individual in

$$N \backslash \{h_1, h_2\} = L \cup (H \backslash \{h_1, h_2\}).$$

Again, as $\#(L) < k$, there exists at least one $h_3 \in H \backslash \{h_1, h_2\}$ such that h_2 likes h_3's widget better:

$$x_{h_3} \succ_{P_{h_2}} x_{h_2},$$

Now, by Lemma 4, h_3 must like x_{h_3} better than x_{h_2} as well,

$$x_{h_3} \succ_{P_{h_3}} x_{h_2}$$

Moreover, by Lemma 4, h_3 must also like x_{h_3} better than x_{h_1},

$$x_{h_3} \succ_{P_{h_3}} x_{h_1},$$

otherwise we could achieve a Pareto improvement by giving x_{h_2} to h_1, x_{h_3} to h_2 and x_{h_1} to h_3.

We may carry out the same arguments inductively until the last element $h_{\#(H)}$ in H is reached within finite steps, as H is finite. By then we have enumerated all the elements in H such that

$$x_{h_{l+1}} \succ_{P_{h_l}} x_{h_l}, \quad \forall l = 1, ..., \#(H) - 1,$$

and moreover, by Lemma 4,

$$x_{h_{\#(H)}} \succ_{h_{\#(H)}} x_{h_l}, \quad \forall l = 1, ..., \#(H) - 1. \tag{19}$$

Define

$$Q_{h_{\#(H)}} := \left\{ z \in M : z \succ_{P_{h_{\#(H)}}} x_{h_{\#(H)}} \right\}.$$

By (19), for any $h^{'} \in H$, $x_{h'} \notin Q_{h_{\#(H)}}$. By Lemma 5, all the widgets in $Q_{h_{\#(H)}}$ needs to be assigned to someone in

$$N \backslash H = L.$$

As $\#(L) < k$ but $Q_{h_{\#(H)}} \geq k$, this is impossible. Hence, we have reached a contradiction. □

A.6 Proof of Theorem 2

Proof. (a). Under the supposition, there exists an allocation $x \in X$ such that

$$R\left(\overline{P}_i, x_i\right) = 1, \quad \forall i \in N,$$

which is the unique Pareto efficient allocation and achieves the best possible ranking profile.

(b). Under $\underline{P}$, any Pareto efficient allocation must consist of the unanimously agreed top N widgets by Lemma 5. Hence, the set of Pareto efficient allocations consists exactly of all permutations of the unanimously agreed top N widgets among the N individuals. Moreover, for any Pareto efficient allocation x, the ranking evaluation vector $R(\underline{P}, x)$ must be a permutation of the vector:

$$\overline{r} := (1, 2, ..., n - 1, N)^{'}.$$

Now, consider any other preference profile P and any Pareto efficient allocation x under P: $x \in PE(P)$. We now seek to prove that there must exist a permutation mapping $\pi : \{1, ..., N\} \rightarrow \{1, ..., N\}$ such that

$$R(P, x) \leq \pi(\overline{r}).$$

Without loss of generality, we may permute individual indexes so that

$$R(P_1, x_1) \leq R(P_2, x_2) \leq ... \leq R(P_N, x_N),$$

and write

$$r := (R(P_1, x_1),\ R(P_2, x_2), ...,\ R(P_N, x_N))'.$$

Applying Lemma 6 with $k = 1$, we have

$$r_1 = 1.$$

Now applying Lemma 6 with $k = 2$, we have

$$\#\{i \in N :\ R(P_i, x_i) \leq 2\} \geq 2,$$

so we have

$$r_2 \in \{1, 2\} \leq 2.$$

Inductively, given that

$$r_i \leq i, \quad \text{for all } i = 1, ..., k-1,$$

we must have by $r_k \leq k$ by Lemma 6(a). Hence, we have

$$r \leq \overline{r}.$$

In summary, there exists a permutation mapping $\pi_x : \{1, ..., N\} \to \{1, ..., N\}$ such that

$$R(P, x) \leq \pi_x(\overline{r}).$$

Now we construct the mapping ψ from $PE(\underline{P})$ to $PE(P)$.

For each $x \in PE(\underline{P})$, there is a unique permutation $(i_1, ..., i_n)$ of N such that

$$R\left(P_{i_k}, \underline{x}_{i_k}\right) = k.$$

We now construct a Pareto efficient allocation y under P in the following way. For i_1, we assign in y to i_1 her favorite widget in M under P_{i_1}, i.e.,

$$y_{i_1} := R^{-1}(P_{i_1}, 1).$$

Trivially,

$$R\left(P_{i_1}, y_{i_1}\right) = 1 \leq 1 = R\left(\underline{P}_{i_1}, x_{i_1}\right).$$

Inductively, given $y_{i_1}, ..., y_{i_{k-1}}$such that

$$R\left(P_{i_h}, y_{i_h}\right) \leq h = R\left(\underline{P}_{i_h}, x_{i_h}\right), \quad \forall h = 1, .., k-1,$$

we assign in x to i_k her favorite widget in $M \backslash \{y_1, ..., y_{k-1}\}$, i.e.,

$$y_{i_k} := \arg \min_{m \in M \backslash \left\{y_{i_1}, ..., y_{i_{k-1}}\right\}} R\left(P_{i_k}, m\right).$$

As $\left\{y_{i_1}, ..., y_{i_{k-1}}\right\}$ involve only $k-1$ widgets, there must exist a widget $m \in M \backslash \left\{y_{i_1}, ..., y_{i_{k-1}}\right\}$ such that $R\left(P_{i_k}, m\right) \leq k$, implying that

$$R\left(P_{i_k}, y_{i_k}\right) \leq k = R\left(\underline{P}_{i_k}, x_{i_k}\right). \tag{20}$$

Induction up to $k = N$ leads to a well-defined allocation y, and we set $y = \psi(x)$.

First, note that $R(P, \psi(x)) \leq R(\underline{P}, x)$ by (20).

Second, y is Pareto efficient under P. To see this, note that each i_k is getting her most preferred widget under P among all widgets *not yet take*n by $i_1, ..., i_{k-1}$, implying that no other allocations could strictly make i_k better off under P without making one of the individuals among $i_1, ..., i_{k-1}$ worse off under P. Hence,

$$\psi\left(PE\left(\underline{P}\right)\right) \subseteq PE(P).$$

Finally, we show that

$$PE(P) \subseteq \psi\left(PE\left(\underline{P}\right)\right).$$

Take any $y \in PE(P)$.

Clearly, there exists some $j_1 \in N$ such that $R\left(P_{j_1}, y_{j_1}\right) = 1$ by Lemma 6.

We now claim that there must exist some $j_2 \in N \backslash \{j_1\}$ such that

$$y_{j_2} = \arg \min_{m \in M \backslash \left\{y_{j_1}\right\}} R\left(P_{j_2}, m\right).$$

Suppose not. Taking any $k_1 \in N \backslash \{j_1\}$, there must exist some $k_2 \in N \backslash \{j_1\}$ such that

$$y_{k_2} = \arg \min_{m \in M \backslash \left\{y_{j_1}\right\}} R\left(P_{k_1}, m\right),$$

which in particular implies that

$$R\left(P_{k_1}, y_{k_2}\right) < R\left(P_{k_1}, y_{k_1}\right).$$

Inductively we can find a sequence of individuals $k_1, k_2, ..., k_n$ in $N\backslash\{j_1\}$ such that

$$R\left(P_{k_h}, y_{k_{h+1}}\right) < R\left(P_{k_h}, y_{k_h}\right) \ \forall h \in \{1, ..., N-1\}.$$

However, as $\#\left(N\backslash\{j_1\}\right) = N-1$, so there must exist two $h_1, h_2 \in \{1, ..., N\}$ such that

$$k_{h_1} = k_{h_2},$$

which contradicts with Lemma4.

Inductively, suppose we have constructed the sequence of individuals $(j_1, ..., j_k)$ such that

$$y_{j_l} = \arg \min_{m \in M\backslash\left\{y_{j_1}, ..., y_{j_{l-1}}\right\}} R\left(P_{j_l}, m\right) \ \forall l = 1, ..., k.$$

We claim that there must exist some $j_{k+1} \in N\backslash\{j_1, ..., j_k\}$ such that

$$y_{j_{k+1}} = \arg \min_{m \in M\backslash\left\{y_{j_1}, ..., y_{j_k}\right\}} R\left(P_{j_{k+1}}, m\right).$$

Suppose not. Taking any $h_1 \in N\backslash\{j_1, ..., j_k\}$, there must exist some $h_2 \in N\backslash\{j_1, ..., j_k\}$ such that

$$y_{h_2} = \arg \min_{m \in M\backslash\left\{y_{j_1}, ..., y_{j_k}\right\}} R\left(P_{h_1}, m\right),$$

which in particular implies that

$$R\left(P_{h_1}, y_{h_2}\right) < R\left(P_{h_1}, y_{h_1}\right).$$

Inductively we can find a sequence of not necessarily distinct individuals $h_1, h_2, ..., h_{n-k+1}$ in $N\backslash\{j_1, ..., j_k\}$ such that

$$R\left(P_{h_l}, y_{h_{l+1}}\right) < R\left(P_{h_l}, y_{h_l}\right) \ \forall h \in \{1, ..., N-k\}.$$

However, as $\#\left(N\backslash\{j_1, ..., j_k\}\right) = N-k$ while , so there must exist two $l_1, l_2 \in \{1, ..., N-k+1\}$ such that

$$h_{l_1} = h_{l_2},$$

which contradicts with Lemma4.

In summary, for each $y \in PE(P)$, we have constructed a permutation $(j_1, ..., j_n)$ of N such that

$$y_{j_k} = \arg \min_{m \in M \setminus \left\{y_{j_1},...,y_{j_{k-1}}\right\}} R\left(P_{j_k}, m\right) \ \ \forall k = 1, ..., N.$$

For this given permutation $(j_1, ..., j_N)$, construct an allocation x by setting

$$x_{j_k} := R^{-1}\left(\underline{P}, k\right),$$

i.e., giving individual j_k the k-th best widget under the common preference $\underline{P}$. Then we have

$$y = \psi(x).$$

Hence, we have constructed an onto mapping $\psi : PE(\underline{P}) \to PE(P)$ such that

$$R(P, \psi(x)) \leq R(\underline{P}, x), \quad \forall x \in PE(\underline{P}).$$

□

A.7 Proof of Theorem 3

Proof. We present the proofs for (b)(c) below, as (a) has been already proved in the main text.

(b) Suppose that there exist some $i \in N$ and two distinct Pareto allocations $x, y \in PE\left(P^{wel}\right)$ such that $x_i = y_i$ and $y_{-i} \succ_{P^{wel}_{i,-i}} x_{-i}$. Then:

$$y \in \left\{z \in X : z_i = x_i \text{ and } z_{-i} \succ_{P^{wel}_{i,-i}} x_{-i}\right\} \neq \emptyset,$$

and (7) is violated, resulting in "$\rhd_X$".

(c) Suppose that $n \geq 3$, $\#\left(PE\left(P^{wel}\right)\right) \geq 2$ and $(P_{i,-i})_{i \in N}$ is strict. We claim that there exists some individual i and two distinct $x, y \in PE\left(P^{wel}\right)$ such that $x_i = y_i$. If this claim is true, as $(P_{i,-i})_{i \in N}$ is strict, then either $x_{-i} \succ_{P_{i,-i}} y_{-i}$ or $y_{-i} \succ_{P_{i,-i}} x_{-i}$ is true, satisfying the condition in (a.i) and our conclusion follows. We now prove the above claim:

By the proof of Theorem (2)(b), for each individual i, there exists at least one Pareto efficient allocation $x \in PE\left(P^{wel}\right)$ such that $R_M\left(P_{i,i}, x_i\right) = 1$. Now, suppose that for each individual i, there exists just a single Pareto efficient allocation $x^{(i)} \in PE\left(P^{wel}\right)$

such that $R_M\left(P_{i,i}, x_i^{(i)}\right) = 1$. For each i and $x^{(i)}$, $x_{-i}^{(i)}$ must be the unique Pareto efficient allocation in $X_{-i}\backslash\left\{z \in X : z_i = x_i^{(i)}\right\}$ according to $(P_{j,j})_{j\neq i}$. This is only possible if $x_j^{(i)}$ is j's favorite widget in $M\backslash\left\{x_i^{(i)}\right\}$ for every $j \neq i$. Hence, $R_M\left(P_{j,j}, x_j^{(i)}\right) \leq 2$ for every $j \neq i$. If $R_M\left(P_{j,j}, x_j^{(i)}\right) = 1$ for all $j \neq i$, then we must have $PE\left(P^{wel}\right) = \left\{x^{(i)}\right\}$, contradicting $\#\left(PE\left(P^{wel}\right)\right) \geq 2$. Hence, there exists at least one individual $j \neq i$ such that $R_M\left(P_{j,j}, x_j^{(i)}\right) = 2$. Now, by Lemma (5), j's favorite widget, denoted x_j^*, must not be available, and in particular we must have $x_i^{(i)} = x_j^*$. Now, as $n \geq 3$, there must exists another individual $k \neq i, j$.

If $x_k^{(i)}$ is individual k's favorite widget, namely x_k^* in M, then there must exist another Pareto efficient allocation $\overline{x} \in PE\left(P^{wel}\right)$ such that $\overline{x}_j = x_j^*$ and $\overline{x}_k = x_k^*$ with $R_M\left(P_{j,j}, \overline{x}_j\right) = R_M\left(P_{j,j}, \overline{x}_k\right) = 1$, i.e., we give i, j's common favorite widget x_j^* to individual j and give individual k's favorite widget to k. Then, we have two distinct $x^{(i)}, \overline{x} \in PE\left(P^{wel}\right)$ such that $x_k^{(i)} = \overline{x}_k = x_k^*$ for individual k, proving our claim above.

Otherwise if $x_k^{(i)}$ is not individual k's favorite widget in M, then individual k's favorite widget, denoted x_k^*, must also be taken by individual i, i.e., $x_i^{(i)} = x_k^*$, and in the meanwhile $R_M\left(P_{k,k}, x_k^{(i)}\right) = 2$. Now, there must exist another Pareto efficient allocation $\overline{x} \in PE\left(P^{wel}\right)$ such that $\overline{x}_j = x_j^*$ and $\overline{x}_k = x_k^{(i)}$ with $R_M\left(P_{j,j}, \overline{x}_j\right) = 1$ and $R_M\left(P_{k.k}, \overline{x}_k\right) = 2$, where we give i, j, k's favorite widget x_j^* to j and give k's second favorite widget $x_k^{(i)}$ to k. Again, we have two distinct allocations $x^{(i)}, \overline{x} \in PE\left(P^{wel}\right)$ such that $x_k^{(i)} = \overline{x}_k = x_k^*$ for individual k, proving our claim above.

□

B Weak Set Order

Alternatively, we could define the partial order on preference profiles by weak set orders on the set of ranking profiles of Pareto efficient allocations under different preference profiles: $P \trianglerighteq^{wso} P'$ whenever

$$R\left(P, PE\left(P\right)\right) \lneqq R\left(P', PE\left(P'\right)\right)$$

in the sense that:

- $\forall x \in PE\left(P\right)$, there exists $y \in PE\left(P'\right)$ such that

$$R\left(P, x\right) \leq R\left(P', y\right).$$

- $\forall y \in PE\left(P^{'}\right)$, there exists $x \in PE(P)$ such that

$$R(P,x) \leq R\left(P^{'},y\right).$$

- At least one inequality is strict.

Corollary 3. *P is $\trianglerighteq^{wso}$-minimal* $\Rightarrow PE(P) = X$.

The converse, however, is not true anymore.

Example. Consider $|X| = N = 6$ and two preference profiles P and P' characterized by

$$R\left(P^{'},X\right) = \begin{pmatrix} 6 & 4 & 1 & 5 & 3 & 2 \\ 5 & 2 & 3 & 6 & 1 & 4 \\ 3 & 4 & 5 & 6 & 2 & 1 \\ 6 & 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} =: \left(r_1^{'},...,r_6^{'}\right),$$

and

$$R(P,X) = \begin{pmatrix} 6 & 5 & 1 & 4 & 3 & 2 \\ 5 & 3 & 2 & 6 & 4 & 1 \\ 2 & 3 & 4 & 6 & 5 & 1 \\ 6 & 5 & 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 6 & 5 & 4 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} =: (r_1,...,r_6),$$

where the i-th column vector corresponds to a ranking evaluation profile of a given allocation x_i and the i-th row vector corresponds to a vector of ranks individual i assigns to all the allocations in X.

It can be checked that $PE\left(P^{'}\right) = PE(P) = X$, but

$$\begin{aligned} r_1^{'} \gneqq r_1, &\quad r_1^{'} \gneqq r_2, \\ r_2^{'} \gneqq r_3, &\quad r \gneqq r_3, \\ r_4^{'} \gneqq r_4, &\quad r_4^{'} \gneqq r_5, \\ r_5^{'} \gneqq r_6, &\quad r_6^{'} \gneqq r_6. \end{aligned}$$

Thus $P \triangleright^{wso} P^{'}$.