

Informal Risk Sharing with Local Information*

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Abstract

This paper considers the effect of local information constraints in risk-sharing networks. We assume individuals only observe the endowment realizations of their neighbors, and bilateral risk-sharing arrangements can only depend on commonly observed information. We derive necessary and sufficient conditions for Pareto efficiency under these constraints in a general setting, and provide an explicit characterization of Pareto efficient arrangements under CARA utilities and normally distributed endowments. With independent endowments, the optimal rule is a local equal-sharing rule. For correlated endowments, the optimal rule relates linearly to a global measure of network centrality. Contrary to other models, individuals with higher centralities are likely to become quasi-insurance providers to more peripheral individuals, attaining more volatile consumption. This framework has important implications for empirical specifications of risk sharing tests. We show that risk-sharing networks generate heterogeneity in insurance opportunities of households that correspond to differences in consumption volatility.

Keywords: social network, risk sharing, Pareto efficiency, local information

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1 Introduction

Informal insurance arrangements in social networks have been shown to play an important role at smoothing consumption in a number of different contexts (Ellsworth 1988, Rosenzweig 1988, Deaton 1992, Paxson 1993, Udry 1994, Townsend 1994, Grimard 1997, Fafchamps and Lund 2003 and Fafchamps and Gubert 2007). A main finding in this literature is that informal insurance achieves imperfect consumption smoothing. There are different theoretical explanations as to why perfect risk sharing is not possible. One leading explanation is the presence of enforcement constraints: since risk-sharing arrangements are informal, they have to satisfy incentive compatibility, implying an upper bound on the amount of transfer that individuals can credibly promise to each other. This type of explanation has been explored in a social network framework by Ambrus, Mobius, and Szeidl (2014).¹

In this paper we explore an alternative explanation featuring local information constraints: individuals can only observe endowment realizations of their direct neighbors, and insurance arrangements between two linked individuals can only be conditioned on local information, consisting of the endowment realizations of their common neighbors (including themselves).² In contrast, existing models of informal risk sharing in networks (Bramoullé and Kranton 2007, Ambrus, Mobius, and Szeidl 2014, Ambrus, Chandrasekhar, and Elliott 2015) assume that any bilateral arrangement between connected individuals can be conditioned on global information, meaning the community’s full set of endowment realizations.³ We find that this explanation generates qualitatively different predictions relative to models of informal insurance with enforcement constraints. Hence, our results can help future empirical work identify

¹See also Karlan, Mobius, Rosenblat, and Szeidl (2009), who investigate enforcement constraints in the case of a single borrowing transaction. There is also an extensive literature on the effects of limited commitment on risk-sharing possibilities for a pair of individuals instead of general networks (Coate and Ravallion 1993, Kocherlakota 1996, Ligon 1998, Fafchamps 1999, Ligon, Thomas, and Worrall 2002, Dubois, Jullien, and Magnac 2008).

²Throughout the paper we maintain the terminology “individuals”, even though in many contexts the relevant unit of analysis is households.

³Bloch, Genicot, and Ray (2008) consider various exogenously-specified transfer rules that do not have to depend on all endowment realizations, but a transfer between two individuals does depend on transfers to or from other individuals - that is on nonlocal information. For example, global equal sharing at a network component level is feasible in the framework of Bloch, Genicot, and Ray (2008), while it is not for general networks in our model. See also Bourlès, Bramoullé, and Perez-Richet (2016), where individuals are motivated to send transfers to their neighbors for explicit altruistic reasons, but like in Bloch, Genicot, and Ray (2008), bilateral transfers depend on transfers among other individuals.

which type of constraint plays the key role in keeping informal insurance arrangements from the efficient frontier.⁴ Most recently, [Milán \(2016\)](#) performs an empirical analysis of this model’s implications using data from indigenous communities in the Bolivian Amazon basin.

There is a line of theoretical literature investigating the effect of imperfect observability of incomes on informal risk sharing arrangements between two individuals in isolation: see for example [Townsend \(1982\)](#), [Thomas and Worrall \(1990\)](#), and [Wang \(1995\)](#). The questions investigated in this literature are fundamentally different from the ones we focus on, mainly because we are interested in questions that are inherently network related.⁵

This paper also examines the link between income and consumption in settings where first-best allocations are not achieved because of incomplete markets. It relates to seminal work by [Attanasio and Pavoni \(2011\)](#) that expands incomplete markets beyond the [Bewley \(1977\)](#) model with a single asset and fixed rate of return, thus allowing for (partial) cross-subsidization of consumption across states. Their friction is a moral hazard on hidden effort and savings, ours an information constraint on endowments along a network.⁶

The current framework also speaks to an ongoing debate in the development literature that emphasizes the importance of appropriately defining individuals’ risk-sharing groups in empirical work. ([Mazzocco and Saini 2012](#), [Angelucci, de Giorgi, and Rasul 2015](#), [Attanasio, Meghir, and Mommaerts 2015](#), [Munshi and Rosenzweig 2016](#)). A general trend in this literature considers alternative sub-groups within communities as the relevant risk-sharing units of individuals (e.g. an individual’s caste or extended family). They argue that classical empirical tests of risk sharing ([Townsend, 1994](#)) must be adapted to accommodate heterogeneity in individuals’ risk sharing communities. However, they only allow for a limited form of heterogeneity in which group membership is mutually exclusive and groups do not interact among

⁴Empirical papers trying to distinguish among different reasons of imperfectness of informal insurance contracts include [Kinman \(2011\)](#) and [Karaivanov and Townsend \(2014\)](#). For an empirical test between full insurance versus informational constraints, see [Ligon \(1998\)](#).

⁵Other differences include that our analysis is static while the above papers are inherently dynamic, and in our paper individuals perfectly observe local information (but not beyond), while in the above papers incomes are not observable even between two connected individuals.

⁶This link between income and the “excess smoothness” of consumption response to permanent income shocks has been explored empirically as well by [Bludell, Pistaferri, and Preston \(2008\)](#) and [Jappelli and Pistaferri \(2010\)](#)

themselves. In this respect, we present a framework that incorporates these partition models, but also allows for much more general forms of interactions in which individuals' sharing groups can overlap in complicated ways along the network. We argue that the current framework has important implications for these new empirical tests of risk-sharing. We show that not defining appropriate local groups biases results, and that standard estimates of risk-sharing tests may be decomposed into an underlying distribution of insurance opportunities that can be interpreted economically in terms of consumption volatility.

The first part of our analysis characterizes Pareto efficient risk-sharing arrangements under local information constraints for general (concave) and possibly heterogeneous utility functions and endowment distributions. Partly as motivation for this exercise, we show that Pareto efficiency in our context (subject to local information constraints) is equivalent to pairwise efficiency, that is the requirement that the risk-sharing agreement between any pair of neighbors is efficient, taking all other agreements between neighbors fixed. This means that any decentralized negotiation procedure that leads to an outcome in which neighbors do not leave money on the table would yield a Pareto efficient risk-sharing arrangement.⁷

In the benchmark model with global information, the necessary and sufficient conditions for Pareto optimality, referred to as the Borch rule (Borch 1962, Wilson 1968) can be derived using standard techniques, and they state that the ratios of any two individuals' marginal utilities of consumptions must be equalized across states. Characterizing the set of Pareto efficient arrangements subject to local information constraints is technically more challenging, as such constraints are intertwined (different pairs of connected individuals are allowed to condition their transfers on different subvectors of the state). Nevertheless, we can generalize the Borch rule to this setting. In particular, a necessary and sufficient condition for Pareto optimality of a risk-sharing arrangement with local information equates the ratios of *expected* marginal utilities of consumption for each linked pair, where expectations are conditional on local states (i.e. on the realizations of commonly observed endowments). As in the context of risk-sharing arrangements with global information, it can be shown

⁷A concrete example for such a negotiation procedure is split the difference negotiations, originally proposed in Stole and Zwiebel (1996) and adopted to the risk sharing context in Ambrus, Chandrasekhar, and Elliott (2015). See also De Fontenay and Gans (2014) for a related pairwise bargaining procedure. In their context pairwise efficiency does not imply Pareto efficiency, but in our context their bargaining procedure would lead to a Pareto efficient outcome.

that for each set of Pareto weights, there is a unique Pareto efficient consumption plan, characterized by the expectational Borch rule.⁸

The generalized Borch rule can be used to verify the Pareto efficiency of consumption plans achieved by candidate transfer agreements in concrete specifications of our model. We provide this characterization for the case of CARA utilities and jointly normally distributed endowments. The characterization is particularly simple for independent endowments: each individual shares her endowment realization equally among her neighbors and herself; on top of that, the arrangement can include state independent transfers, affecting the distribution of surplus but not the aggregate welfare. This type of transfer arrangement, which we refer to as the local equal sharing rule, was considered as an ad hoc sharing rule in [Gao and Moon \(2016\)](#). Our result provides micro-foundations for the rule, in the context of CARA utilities and independent normally distributed endowments. The rule is particularly simple in that bilateral transfers are linear in endowment realizations and they only depend on the pair’s endowment realizations, not on those of common neighbors.

For the more general case of correlated endowment realizations in the CARA-normal setting, we show that efficient risk-sharing can still be achieved by transfers that are linear in endowment realizations and strictly bilateral (i.e. only contingent on the endowment realizations of the pair involved). Moreover, we derive a closed form solution for these efficient transfers for any given network structure. In contrast to the local equal sharing rule that obtains in the case of independent endowments, we find that if individuals i and j are linked, increasing the exposure of i to transfers from non-common neighbors increases the share of i ’s endowment realization transferred to j , relative to local equal sharing, and decreases the share of j ’s endowment realization transferred to i . These correction terms, which are complicated functions of the network structure, take into account that more centrally located individuals are more exposed to the common shock component, and optimally correct for this discrepancy. These correction effects can be summarized in two different measures of an individual’s network centrality. These measures depend on the individual’s network position, as well as the correlation coefficient between endowment realizations. In particular, one centrality measure is related to the fraction of her own endowment

⁸Just like in the standard setting with risk-sharing arrangements that can be conditioned on global information, for arrangements that can only be conditioned on local information it also holds that the set of Pareto efficient risk-sharing arrangements are equivalent to the set of solutions for a utilitarian planner’s problem, for different weights.

realization an individual keeps, relative to the transfers sent to other neighbors, while the other centrality measure is related to the fraction of the neighbors’ endowment the individual receives. In general, more central individuals, according to these measures, transfer a higher share of their endowment realizations to less central neighbors, and conversely take a smaller fraction of neighbors’ endowment realizations.

Since the centrality measures can be computed from observable variables, the above predictions can be tested empirically. In fact, [Milán \(2016\)](#) tests the pairwise transfer scheme predicted by local information constraints against the observed exchanges of food between households of the Bolivian Amazon. The analysis shows that the network-based transfers under local information are good predictors of the pairwise sharing behavior of these aboriginal communities, and that local information constraints can adequately account for the departure of these communities from the efficient benchmark.

Even with the above correction terms relative to local equal sharing, more central individuals end up with a higher consumption variance because they serve as quasi insurance providers to more peripheral neighbors. For a fixed set of welfare weights, they are compensated for this service through higher state-independent transfers (“insurance premium”). This is contrary to the predictions from models with enforcement constraints, like AMS, in which more centrally connected individuals are better insured (end up with smaller consumption variance) because for typical endowment realizations they end up on larger “risk-sharing islands.” While the centrality measures delivered by our model are not equivalent to more standard notions of centrality, still we show that, for typical networks, the contrast between the predictions of our model and the AMS model remains when we use degree or eigenvector centrality as a substitute. We show this through simulations, using network data from two different data sources from Indian villages.⁹ With simulated endowment realizations, the AMS model produces a negative correlation between either degree or eigenvalue centrality and consumption variance, more starkly for relatively tight capacities, while the optimal risk-sharing arrangements in our model yield positive correlations.

Our model can also explain why informal insurance might perform badly in one setting but quite well in another, despite similarities in network structure and in the average correlation across endowments. In particular, with local information con-

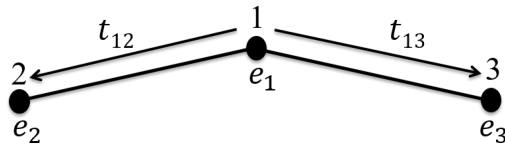
⁹The data was provided to us by Erica Field and Rohini Pande, and by Abhijit Banerjee, Arun Chandrasekhar, Esther Duflo and Matthew Jackson.

straints, high correlation between neighboring individuals’ endowments substantially hurts risk-sharing efficiency. We demonstrate this property in a circle network, under the assumption that correlations between endowments geometrically decay with distance. Keeping the same level of insurable risk under the global information risk-sharing arrangements benchmark, and focusing on the case of a large number of individuals and highly correlated endowment realizations, we show that risk-sharing arrangements with local information improve very little over autarky under the decaying correlation structure, while fairly good risk-sharing can be achieved under the symmetric correlation structure (correlation not depending on social distance). This might help explain why, even though empirical research has found that informal insurance works well in many contexts, [Kazianga and Udry \(2006\)](#) found a setting in which informal insurance does not seem to help, and [Goldstein, de Janvry, and Sadoulet \(2001\)](#) found that certain types of shocks are not well insured through informal risk sharing. Moreover, we think that a setting in which correlations across endowments decay over the network distance captures a very realistic phenomenon in risk sharing contexts, where individuals that are “socially close” will tend to engage in similar productive activities.¹⁰

The rest of this paper is organized as follows. In Section 2, we use a simple example to illustrate the differences between risk sharing with global and local information. In Section 3, we present our general model and derive a set of necessary and sufficient conditions for Pareto efficient transfer arrangements subject to the local information constraints. In Section 4 we explicitly characterize constrained Pareto efficient transfer arrangements in settings with CARA utilities and jointly normally distributed endowments, and examine their properties. In Section 5 we outline the theory’s implications for empirical tests of risk-sharing, and we show how standard estimates can be decomposed into underlying heterogeneity and interpreted economically. In Section 6 we discuss a number of extensions and generalizations. Section 7 concludes. Proofs and lemmas are provided in the Appendix A and supplementary materials are available in Appendix B.

¹⁰See for instance, [Hooper \(2011\)](#), [Fafchamps and Lund \(2003\)](#), etc.

Figure 1: Simple 3-Individual Network



2 Example: Three Individuals in a Line Network

2.1 Specification

Before investigating general network structures, we first consider the simplest non-trivial network, where three individuals, denoted by 1, 2 and 3, are minimally connected. In particular, individual 1 is linked with individual 2 and individual 3, but individual 2 is not linked with individual 3. Despite its simplicity, this example provides some useful insights on how local information constraints affect efficient risk-sharing arrangements.

We assume that individuals have homogeneous CARA preferences of the form $u(x) = -\exp(-rx)$. Furthermore, assume that endowments $e_1, e_2, e_3 \sim_{iid} \mathcal{N}(0, \sigma^2)$. Only linked individuals may enter into risk-sharing arrangements to mitigate endowment risks. Let t_{12} denote the net ex post transfer from individual 1 to individual 2, t_{13} the net transfer from individual 1 to individual 3. Let x_1, x_2, x_3 denote the final consumption to individuals after the transfers are implemented, i.e., $x_1 = e_1 - t_{12} - t_{13}$, $x_2 = e_2 + t_{12}$ and $x_3 = e_3 + t_{13}$.

Below we compare Pareto efficient transfer rules in two cases: under risk-sharing arrangements between neighbors that can condition transfers on everyone's endowment realization (global information), and under risk-sharing arrangements that can only condition on endowment realizations of the two individuals forming the arrangement (local information).

2.2 Risk-sharing Arrangements with Global Information

First we consider the benchmark case when bilateral risk-sharing arrangements can be conditioned on global information, that is on all three individuals' endowment realizations, so that t_{12}, t_{13} can be arbitrary functions of the endowments e_1, e_2, e_3 . Standard arguments (see [Wilson, 1968](#)) establish that Pareto efficient transfer rules

t_{12}, t_{13} are the ones maximizing the social planner's problem:

$$\mathbb{E} \left[\sum_{i=1}^3 \lambda_i u(x_i) \right] = \mathbb{E} [\lambda_1 u(e_1 - t_{12} - t_{13}) + \lambda_2 u(e_2 + t_{12}) + \lambda_3 u(e_3 + t_{13})],$$

for some $\lambda_1, \lambda_2, \lambda_3 \in (0, 1)$ s.t. $\lambda_1 + \lambda_2 + \lambda_3 = 1$. By the well-known Borch rule (Borch 1962, Wilson 1968), the necessary and sufficient conditions for optimality are:

$$\lambda_1 u'(e_1 - t_{12} - t_{13}) = \lambda_2 u'(e_2 + t_{12}) = \lambda_3 u'(e_3 + t_{13}) \quad \forall e_1, e_2, e_3.$$

With CARA utility, this yields $t_{12}(e_1, e_2, e_3) = \frac{1}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3 - \frac{1}{3r} \ln(\lambda_2^2/\lambda_1\lambda_3)$ and similarly for t_{13} , leading to the the final consumption plan:

$$\begin{cases} x_1 = \frac{1}{3}(e_1 + e_2 + e_3) + \frac{1}{3r} \ln(\lambda_2\lambda_3/\lambda_1^2) \\ x_2 = \frac{1}{3}(e_1 + e_2 + e_3) + \frac{1}{3r} \ln(\lambda_1\lambda_3/\lambda_2^2) \\ x_3 = \frac{1}{3}(e_1 + e_2 + e_3) + \frac{1}{3r} \ln(\lambda_1\lambda_2/\lambda_3^2). \end{cases} \quad (1)$$

That is, Pareto efficient risk-sharing arrangements in every state divide total realized endowments equally among individuals, and the equal division is then corrected by state-independent transfers that achieve the welfare weights.

2.3 Risk-sharing Arrangements with Local information

Suppose now that endowment realizations are only locally observable and verifiable, so that the transfers t_{12}, t_{13} in the risk-sharing arrangements can be contingent on local information only, that is: $t_{12} = t_{12}(e_1, e_2)$, $t_{13} = t_{13}(e_1, e_3)$.

Achieving consumption plans on the Pareto frontier, given by (1), is impossible subject to these local information constraints. However, a necessary condition for a transfer arrangement to be socially optimal¹¹ is that, for any given realization of e_1 and e_2 , t_{12} should maximize $\lambda_1 u(e_1 - t_{12} - t_{13}) + \lambda_2 u(e_2 + t_{12})$, given the distribution of e_3 conditional on e_1 and e_2 , and the implied distribution of consumption levels (net of t_{12}) induced by t_{13} . In short, given t_{13} , t_{12} should maximize the planner's welfare function:

$$\max_{t_{12}} \int [\lambda_1 u(e_1 - t_{12} - t_{13}) + \lambda_2 u(e_2 + t_{12})] f_{3|12}(e_3) de_3$$

¹¹We show in Section 3 that this condition is actually also sufficient.

The necessary and sufficient FOC for this maximization problem is:

$$\lambda_1 \mathbb{E} \left[u' (e_1 - t_{12} - t_{13} (e_1, e_3)) \middle| e_1, e_2 \right] = \lambda_2 u' (e_2 + t_{12}),$$

and similarly for t_3 given t_2 . Solving this system of two integral equations, we obtain the following transfer rule

$$t_{12}(e_1, e_2) = \frac{1}{3}e_1 - \frac{1}{2}e_2 - \frac{1}{24}r\sigma^2 - \frac{1}{3r} \ln (\lambda_1 \lambda_3 / \lambda_2^2) \quad (2)$$

and similarly for $t_{13}(e_1, e_3)$. Notice the transfers can be decomposed into three parts. The first part, $\frac{1}{3}e_1 - \frac{1}{2}e_2$, corresponds to the “local equal sharing rule”, which is the local variant of the equal sharing rule. It implies that individual i 's endowment e_i is equally shared by i and i 's neighbors, i.e., $t_{ij} = \frac{e_i}{d_i+1} - \frac{e_j}{d_j+1}$. The second part of the equations in (2), $-\frac{1}{24}r\sigma^2$, corresponds to a state-independent transfer that can be regarded as the “insurance premium” paid by the “net insurance purchaser” to the “net insurance provider”. In this case, as the final consumption are

$$\begin{cases} x_1 = \frac{1}{3}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{1}{12}r\sigma^2 + \frac{1}{3r} \ln (\lambda_1^2 / \lambda_2 \lambda_3), \\ x_2 = \frac{1}{3}e_1 + \frac{1}{2}e_2 - \frac{1}{24}r\sigma^2 + \frac{1}{3r} \ln (\lambda_2^2 / \lambda_1 \lambda_3), \\ x_3 = \frac{1}{3}e_1 + \frac{1}{2}e_3 - \frac{1}{24}r\sigma^2 + \frac{1}{3r} \ln (\lambda_3^2 / \lambda_1 \lambda_2), \end{cases}$$

individual 1 takes extra risk exposure $\frac{1}{3}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3$ in comparison to individuals 2 and 3, $\frac{1}{3}e_1 + \frac{1}{2}e_2$ or $\frac{1}{3}e_1 + \frac{1}{2}e_3$. Hence, individual 1 is rewarded the certainty equivalent (CE) for her intermediary role in risk sharing. The third part of the equations in (2), $-\frac{1}{3r} \ln (\lambda_1 \lambda_2 / \lambda_3^2)$, redistributes wealth according to the welfare weights assigned to different individuals (it is zero when $\lambda_1 = \lambda_2 = \lambda_3$).

To evaluate the welfare loss associated with risk-sharing arrangements that condition only on local information, since social welfare is a linear, strictly decreasing function of total variances under CARA utilities and normal endowments, we can simply compare the total variances of final consumption. With global information, the sum of consumption variances is: $TVar_G = 3 \cdot Var \left[\frac{1}{3} (e_1 + e_2 + e_3) \right] = \sigma^2$. With bilateral risk-sharing arrangements subject to the local information constraints, the sum of consumption variances increases to: $TVar_L = Var \left[\frac{1}{3}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 \right] + Var \left[\frac{1}{3}e_1 + \frac{1}{2}e_2 \right] + Var \left[\frac{1}{3}e_1 + \frac{1}{2}e_3 \right] = \frac{4}{3}\sigma^2$. Hence the welfare loss arising from local information constraints is $\frac{1}{3}\sigma^2$ in the above example.

3 General Conditions for Pareto Efficiency

Before we proceed to our main analysis, we introduce some notations. Let $N = \{1, 2, \dots, n\}$ be a finite set of individuals and let G be the adjacency matrix of a network structure on N . A pair of individuals i, j are linked if $G_{ij} = 1$, and by convention, $G_{ii} = 0$. Throughout the paper we assume, without loss of generality, that G represents a connected network.¹² Denote the neighborhood of i by $N_i := \{j \in N : G_{ij} = 1\}$ and the extended neighborhood of i by $\bar{N}_i := N_i \cup \{i\}$. The degree of individual i is defined as $d_i := \#(N_i)$, the number of individuals to whom i is linked. The state of the world is defined as the vector of endowment realizations $e \equiv (e_i)_{i \in N} \in \Omega \equiv \mathbb{R}^n$, and its distribution is given by a probability measure \mathbb{P} on $(\Omega, \mathcal{B}(\Omega))$. We assume that the distribution of e has finite expectation.

A central assumption in our paper is that individuals can only observe the endowment realizations of their direct neighbors. Define $N_{ij} := N_i \cap N_j$ and $\bar{N}_{ij} := \bar{N}_i \cap \bar{N}_j$. Let $I_i(e) := (e_j)_{j \in \bar{N}_i}$ be the information vector of i , and $I_{ij}(e) := (e_k)_{k \in \bar{N}_{ij}}$ be the common information vector of a linked pair ij . We also refer to I_{ij} as the local state for ij .

We assume that only linked pairs of individuals can engage in informal risk sharing directly. An ex ante risk-sharing arrangement between linked individuals i and j is a net transfer rule $t_{ij} : \Omega \rightarrow \mathbb{R}$, which prescribes a net amount of $t_{ij}(e)$ to be transferred from i to j at each realized state e . We assume that linked pairs can only condition the net transfer on their their ex post local common information, i.e., their local state. Formally, we require that $t_{ij} : \Omega \rightarrow \mathbb{R}$ be $\sigma(I_{ij})$ -measurable, where $\sigma(I_{ij})$ denotes the sub- σ -algebra induced by I_{ij} . By definition, $t_{ij}(e) = -t_{ji}(e)$ for every $e \in \Omega$ and linked $i, j \in N$. We refer to the profile of ex ante risk-sharing arrangements t_{ij} between all pairs of linked individuals as a transfer arrangement t .

Let \mathcal{T} denote the set of all admissible transfer arrangements $t : \Omega := \mathbb{R}^n \rightarrow \mathbb{R}^{\sum_{i \in N} d_i}$ that are only contingent on the local state for each linked pair:

$$\mathcal{T} := \left\{ t : \Omega \rightarrow \mathbb{R}^{\sum_{i \in N} d_i} \left| \begin{array}{l} \forall i, j \text{ s.t. } G_{ij} = 1, \quad \text{and } t_{ij}(e) + t_{ji}(e) = 0, \forall e \in \Omega, \\ \text{and } \mathbb{E}[t_{ij}] \text{ is finite.} \end{array} \right. \right\}$$

¹²Otherwise we may analyze each component separately.

Define $\langle s, t \rangle := \mathbb{E} \left[\sum_{G_{ij}=1} s_{ij}(e) t_{ij}(e) \right]$ for any $s, t \in \mathcal{T}$. It follows that $\langle \cdot, \cdot \rangle$ is an inner product and thus \mathcal{T} is a well-defined inner product space (see Lemma 1 in B.1 for a formal proof). We slightly abuse notations by treating each element in \mathcal{T} as an equivalent class of transfer arrangements that are indistinguishable under the norm induced by $\langle \cdot, \cdot \rangle$. Throughout the paper, we write “ $s = t$ ” to mean “ $\langle s - t, s - t \rangle = 0$ ”.

Given a transfer arrangement $t \in \mathcal{T}$, we define the final consumption plan induced by t as $x^t : \Omega \rightarrow \mathbb{R}^n$ with $x_i^t(e) := e_i - \sum_{h \in N_i} t_{ij}(e)$. Individuals derive utilities from their own final consumption, and we assume that they have a strictly concave and twice differentiable utility function u , with $u' > 0$ and $u'' < 0$.

To characterize the set of Pareto efficient transfers under the local information constraint, we solve the following problem:

$$\max_{t \in \mathcal{T}} J(t) := \mathbb{E} \left[\sum_{k \in N} \lambda_k u \left(e_k - \sum_{h \in N_k} t_{kh} \right) \right] \quad (3)$$

Recall that both e and t are assumed to have finite expectation. As u is strictly concave, by Jensen’s inequality, we conclude that $\mathbb{E} \left[u \left(e_k - \sum_{h \in N_k} t_{kh} \right) \right] < \infty$ for all $k \in N$, so the social welfare function $J : \mathcal{T} \rightarrow \mathbb{R} \cup \{-\infty\}$ is well defined on \mathcal{T} .

The following proposition provides a formal characterization of the solution to the maximization problem above. Since the transfer rule t_{ij} is restricted to be measurable with respect to $\sigma(I_{ij})$, we slightly abuse notations and write it as $t_{ij} : \mathbb{R}^{d_{ij}+2} \rightarrow \mathbb{R}$ where $d_{ij} + 2 = \dim(I_{ij})$. We denote the distribution of I_{ij} on $\mathbb{R}^{d_{ij}+2}$ by $\mathbb{P}I_{ij}^{-1}$.

Proposition 1. *A profile of $t \in \mathcal{T}$ solves (3) if and only if it simultaneously solves the $\sum_{i \in N} d_i$ optimization problems in the form of (4) at $\mathbb{P}I_{ij}^{-1}$ -almost all possible local states of the linked pair: $\forall i, j$ s.t. $G_{ij} = 1$, for $\mathbb{P}I_{ij}^{-1}$ -almost all $\tilde{I}_{ij} \in \mathbb{R}^{d_{ij}+2}$,*

$$t_{ij}(\tilde{I}_{ij}) \in \arg \max_{t_{ij} \in \mathbb{R}} \mathbb{E} \left[\begin{array}{l} \lambda_i u_i \left(e_i - \tilde{t}_{ij} - \sum_{h \in N_i \setminus \{j\}} t_{ih}(e) \right) \\ + \lambda_j u_j \left(e_j + \tilde{t}_{ij} - \sum_{h \in N_j \setminus \{i\}} t_{jh}(e) \right) \end{array} \middle| I_{ij}(e) = \tilde{I}_{ij} \right] \quad (4)$$

Proposition 1 is an intuitive result, and the analogue of it is easy to show when risk-sharing arrangements can be conditioned on everyone’s income realization, as in Wilson (1968). In that case, for each possible realizations of e , we may freely choose $(t_{ij}(e))_{G_{ij}=1}$, a finite dimensional vector, to maximize the weighted sum of utilities. This enables for standard finite dimensional optimization techniques, leading to first

order conditions for optimum that just connect ratios of marginal utilities of consumption for two individuals in two different states. In contrast, the intertwined local information structure induced by the local information constraint makes the optimization problem fundamentally infinite-dimensional. State-by-state optimization is no longer feasible: taking for example again the network depicted in Figure 1, we see that the local state for the pair 12 is given by $I_{12}(e) = (e_1, e_2)$, and the transfer t_{12} must be constant across e_3 . As the value of t_{13} can vary continuously as a function of e_3 , the efficient t_{12} must be optimal in expectation with respect to the distribution of t_{13} conditional on $I_{12}(e)$. Similarly, t_{13} must be optimal with respect to the conditional distribution of t_{12} on each $I_{13}(e)$. As the realizations of I_{12} and the realizations of I_{13} induce two different uncountable partitions of the state space, with the transfers being constant on respective slices of the partitions, we can no longer carry out state-by-state optimization at each e , but have to optimize the transfers at all states simultaneously.

The proof of Proposition 1 formally establishes this using mathematical results from convex optimization in normed spaces. We show that the objective function is strictly concave and Gâteaux-differentiable on \mathcal{T} . Then, a sufficient and necessary condition for optimality is given by the first Gâteaux-derivative of the objective being the zero function, which is the functional generalization of the usual first-order condition for optimality.

Proposition 1 establishes an equivalence between Pareto-efficient risk-sharing arrangements subject to local information constraints and stable outcomes of decentralized bilateral risk sharing arrangements between neighbors subject to the same constraints. In particular, in our context any bargaining procedure that leads to an agreement between any two neighboring agents that efficient for the pair (does not leave surplus on the table), given other agreements, results in a Pareto efficient outcome at the social level. Notice that in problem (4), at each I_{ij} , the choice of t_{ij} affects the expected utilities of only i and j , so each optimization problem in (4) can be reinterpreted as the surplus maximization problem jointly solved by the linked pair ij , given the transfer rules chosen by other linked pairs. Hence Proposition 1 provides us a further motivation for investigating Pareto efficient risk sharing arrangements subject to local information constraints, by implying that these are exactly the possible outcomes resulting from decentralized negotiation procedures satisfying the weak requirement that neighboring agents end up with agreements that are efficient at the

pair level.

The next result establishes that while in general there can be multiple transfer profiles satisfying the conditions for optimality (4), they all imply the same consumption plan in all states.

Proposition 2. *All profiles of transfers $t \in \mathcal{T}$ that solve (3) lead to (\mathbb{P} -almost) the same consumption plan x .*

By Proposition 2, if we can find a profile of transfers so that the induced consumption plan satisfy (4), then it must correspond to a Pareto efficient risk-sharing arrangement.

For simplicity, below we will denote the conditional expectation operator $\mathbb{E}[\cdot | I_{ij}]$ by $\mathbb{E}_{ij}[\cdot]$. In observation of Proposition 1 and Proposition 2, we may express the necessary and sufficient condition for Pareto efficiency as a requirement on the ratio of conditional expected marginal utilities given by the next Corollary.

Corollary 1. *A profile of transfers t is Pareto efficient if and only if the ratio of the expected marginal utilities conditional on all local states is constant: for every $i, j \in N$ s.t. $G_{ij} = 1$,*

$$\frac{\mathbb{E}_{ij} [u'_i(x_i^t)]}{\mathbb{E}_{ij} [u'_j(x_j^t)]} = \frac{\lambda_j}{\lambda_i}. \quad (5)$$

This extends the Borch rule (Borch 1962, Wilson 1968) for Pareto efficient risk-sharing arrangements to settings with local information constraints. As opposed to the case when risk-sharing arrangements can be conditioned on the endowment realizations of all players, the ratio of expected marginal utilities of consumptions among individuals do not have to be equal state by state, they only have to be equal between linked individuals in expectation, conditional on common information.

4 Efficient Risk-sharing Arrangements under the CARA-Normal Setting

In this section we investigate Pareto efficient risk-sharing arrangements, subject to local information constraints, under the assumption of CARA utilities and jointly normally distributed endowments.

Assumption 1. Throughout the subsequent sections we assume that individuals have homogeneous CARA utility functions $u(x) = -\exp(-rx)$, where $r > 0$ is the coefficient of absolute risk aversion. The vector of endowments $(e_i)_{i \in N}$ follows a multivariate normal distribution, $e \sim \mathcal{N}(\mathbf{0}, \sigma^2 \Sigma)$ with, for some $\rho \in [-\frac{1}{n-1}, 1]$,¹³

$$\Sigma := \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \rho & 1 \end{pmatrix}.$$

4.1 Independent Endowments

We first analyze the case where endowments are independent, i.e., $e \sim \mathcal{N}(\mathbf{0}, \sigma^2 \cdot \mathbf{I}_n)$.

We use a guess and verify approach, and postulate that a linear transfer scheme, that is a scheme for which the transfer between any two connected individuals is a linear function of endowment realizations in the pair's joint information set, can achieve any Pareto efficient risk-sharing arrangement. Below we verify that the candidate linear risk-sharing arrangement is globally optimal subject to local information constraints using the expectational Borch rule (5).

Given a linear transfer scheme, the final consumptions, conditional on I_{ij} , also follow normal distribution, so $\mathbb{E}_{ij}[u'_i(x_i)] = r \exp[-r(\mathbb{E}_{ij}[x_i] - \frac{1}{2}r \text{Var}_{ij}[x_i])]$. Define the conditional certainty equivalent $CE(x_i | I_{ij}) := \mathbb{E}_{ij}[x_i] - \frac{1}{2}r \text{Var}_{ij}[x_i]$. Then (5) can then be rewritten as

$$CE(x_i^* | I_{ij}) - \frac{1}{r} \ln \lambda_i = CE(x_j^* | I_{ij}) - \frac{1}{r} \ln \lambda_j. \quad (6)$$

The profile of transfer schemes t achieves Pareto efficiency if and only if (6) holds for every pair of ij such that $G_{ij} = 1$, i.e., the difference in the conditional certainty equivalents is constant at each local state for a linked pair.

We say a profile of transfer rules is *strictly bilateral* if t_{ij} is $\sigma(e_i, e_j)$ -measurable.

Proposition 3. *Given any profile of positive welfare weights $(\lambda_i)_{i \in N}$, there always exists a strictly bilateral Pareto efficient profile of transfer rules in \mathcal{T} in the form of*

¹³ $-\frac{1}{n-1}$ is the lower bound for a global pairwise correlation in a n -person economy; mathematically, it is the smallest ρ such that the variance-covariance matrix is positive semi-definite. For any $\rho \in [-\frac{1}{n-1}, 1]$, the variance-covariance matrix is positive semi-definite.

$t_{ij}^*(e_i, e_j) := \frac{e_i}{d_i+1} - \frac{e_j}{d_j+1} + \mu_{ij}^*$, for some $\mu_{ij}^* \in \mathbb{R}$, for each linked pair ij .

Recall that, by Proposition 2, the Pareto efficient consumption plan is unique. But for general networks, there might be multiple risk-sharing arrangements that are Pareto efficient. In particular, superfluous transfers, either state-dependent or state-independent, may be freely added to a cycle of individuals in the network without changing the final consumptions. Therefore, in general the transfer scheme achieving a Pareto efficient risk-sharing arrangement is not unique. In Appendix B.2 we show that for tree networks the linear transfer scheme featured in Proposition 3 is the unique transfer arrangement that achieves a given Pareto efficient risk-sharing arrangement.

The efficient transfer $t_{ij}^*(e_i, e_j)$ subject to the local information constraint is composed of two parts: the state-contingent “local equal sharing rule” and the state-independent “insurance premium”. Furthermore, the transfers between two connected individuals ascribed by the linear transfer scheme in Proposition 3 only depend on endowment realizations of the two of them, not of their common neighbors. That is, only bilateral information is required for efficient risk sharing with local information. Also, ex ante two linked individuals ij only need knowledge of the local network structure (in particular d_i and d_j) to compute and contract on the socially optimal transfer rule t_{ij}^* .

4.2 Correlated Endowments

We now turn to the case of correlated endowments with $\rho \neq 0$. To maintain analytical tractability, in Assumption 1 we assume a symmetric correlation structure, where any two individuals’ endowments have a constant pairwise correlation coefficient $\rho \in [-\frac{1}{n-1}, 1]$. Equivalently, we are assuming that each individual’s endowment can be decomposed additively into two independent components: a common shock and a idiosyncratic shock, i.e., $e_i = \sqrt{\rho}\tilde{e}_0 + \sqrt{1-\rho}\tilde{e}_i$, with $(\tilde{e}_k)_{k=0}^n \sim_{iid} \mathcal{N}(0, \sigma^2)$.

Below we show that any Pareto efficient risk-sharing arrangement can be achieved through a linear transfer arrangement. We first consider the case of minimally-connected networks, for notational simplicity, and to develop intuition. Notice that, under minimal connectedness, $I_{ij} = (e_i, e_j)$, so transfer t_{ij} must be strictly bilateral.

Then, the local FOC for optimality can be written as

$$\begin{aligned}
t_{ij} = & \frac{1}{2}e_i - \frac{1}{2}e_j - \frac{1}{2r} \ln \mathbb{E} \left[\exp \left(r \sum_{k \in N_i \setminus \{j\}} t_{ik}(e_i, e_k) \right) \middle| e_i, e_j \right] \\
& + \frac{1}{2r} \ln \mathbb{E} \left[\exp \left(r \sum_{k \in N_j \setminus \{i\}} t_{jk}(e_j, e_k) \right) \middle| e_i, e_j \right] + \frac{1}{2r} \ln \frac{\lambda_j}{\lambda_i}
\end{aligned} \tag{7}$$

Postulating a linear transfer scheme of the form, $t_{ij}(e_i, e_j) = \alpha_{ij}e_i - \alpha_{ji}e_j + \mu_{ij} \forall G_{ij} = 1$, we can substitute the postulated linear forms of t_{ik} into (7) and obtain

$$\begin{aligned}
t_{ij} = & \frac{1}{2}e_i - \frac{1}{2}e_j - \frac{1}{2} \sum_{k \in N_i \setminus \{j\}} \alpha_{ik}e_i - \frac{1}{2r} \ln \mathbb{E} \left[\exp \left(-r \sum_{k \in N_i \setminus \{j\}} \alpha_{ki}e_k \right) \middle| e_i, e_j \right] \\
& + \frac{1}{2} \sum_{k \in N_j \setminus \{i\}} \alpha_{jk}e_j + \frac{1}{2r} \ln \mathbb{E} \left[\exp \left(-r \sum_{k \in N_j \setminus \{i\}} \alpha_{kj}e_k \right) \middle| e_i, e_j \right] \\
& - \frac{1}{2} \sum_{k \in N_i \setminus \{j\}} \mu_{ik} + \frac{1}{2} \sum_{k \in N_j \setminus \{i\}} \mu_{jk} + \frac{1}{2r} \ln \frac{\lambda_j}{\lambda_i}
\end{aligned}$$

Given the assumed correlation structure of endowments, the conditional distribution $e_k |_{e_i, e_j} \sim \mathcal{N} \left(\frac{\rho}{1+\rho} (e_i + e_j), \frac{1+\rho-2\rho^2}{1+\rho} \cdot \sigma^2 \right)$. We can explicitly derive the conditional expectation terms in the above formula and, after collecting terms and reconciling with the postulated formula for t_{ij} , we arrive at the following system of equations:

$$\alpha_{ij} = \frac{1}{2} \left[1 - \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} + \frac{\rho}{1+\rho} \left(\sum_{k \in N_i \setminus \{j\}} \alpha_{ki} - \sum_{k \in N_j \setminus \{i\}} \alpha_{kj} \right) \right] \quad \forall ij \text{ s.t. } G_{ij} = 1 \tag{8}$$

Rigorously there should be another set of equations that verify the guess for the state-independent constant transfers μ , which in general involve both α and μ . However, Lemma 6 in the Appendix implies that, given any admissible α , there exist some μ such that (α, μ) satisfies the set of verification equations for the constant transfers. Hence, it is system (8), which involves only α , that constitutes the essential condition for Pareto efficiency. We therefore omit the conditions on μ and delay our discussion about state-independent transfers to Section 6.3.

In equation (8), the net transferred share α_{ij} of e_i from i to j is given by the half

of the “remaining share” after deducting the transfers to i ’s other neighbors $N_i \setminus \{j\}$, corrected by an adjustment for inflows of non-local endowments. The $\frac{1}{2}$ multiplier is analogous to the equal sharing rule in the independent endowments case, but last term in the square brackets is new.¹⁴ We refer to it as an informational effect, for the following reason. $\sum_{k \in N_i \setminus \{j\}} \alpha_{ki}$ is the sum of i ’s shares of i ’s other neighbors’ endowments $(e_k)_{k \in N_i \setminus \{j\}}$, and the conditional expectation of each k ’s endowment changes linearly with the realization of e_i by a factor of $\frac{\rho}{1+\rho}$. Similarly, $\sum_{k \in N_j \setminus \{i\}} \alpha_{kj}$ is the sum of j ’s shares of j ’s other neighbors’ endowments $(e_k)_{k \in N_j \setminus \{i\}}$, and the conditional expectation of each k ’s endowment also changes linearly with the realization of e_i by a factor of $\frac{\rho}{1+\rho}$. Due to the symmetric correlation structure, the realization of e_i provides the same amount of local information about all non-local endowment realizations e_k for $k \notin \bar{N}_{ij}$, and thus its informational effect can be calculated as a simple net summation of endowment shares. As a higher realized e_i predicts that both i and j are more likely to obtain higher amounts of inflows from uncommon neighbors, this commonly recognized information can be used by the pair ij to (imperfectly) share the non-local risk exposures.¹⁵ After pooling the conditional expectations of non-local inflows, i and j again share the *remaining* shares of e_i and e_j equally. It is worth pointing out that i carries out this kind of “equal sharing” with all her neighbors, and the inflow/outflow shares (α_{ij}) must make all the sharing simultaneously equal (in expectation).

Hence, the $\sum_{i \in N} d_i$ -dimensional vector $(\alpha_{ij})_{G_{ij}=1}$ must solve the system of $\sum_{i \in N} d_i$ linear equations defined by (8). No more linear restrictions need to be imposed on $(\alpha_{ij})_{G_{ij}=1}$, because, for each i , i herself absorbs $1 - \sum_{j \in N_i} \alpha_{ij}$ of her own endowment so that e_i is fully shared within i ’s extended neighborhood. It can be shown that this system has a solution, but we leave this to be established later, for general networks.

For general network structure, the analysis is very similar to the above, but there are several complications. As $I_{ij} = (e_i, e_j, e_{N_{ij}})$, the transfer rule t_{ij} can be contingent on $e_{N_{ij}} := (e_k)_{k \in N_{ij}}$ in addition to e_i, e_j . Furthermore, as the knowledge of the ex post realization of $e_{N_{ij}}$ brings in extra information about the distribution of non-

¹⁴This term disappears when $\rho = 0$.

¹⁵To be precise, by “inflow” we mean the undertaking of a share of someone else’s income endowment, which may be positive or negative; by “outflow” we mean the distribution of a share of one’s own endowment to someone else, which may also be positive or negative. In particular, a negative inflow is not the same as an outflow. Instead, i ’s inflow from j is the same as j ’s outflow to i .

local endowment realizations, Pareto efficiency requires that t_{ij} be contingent on $e_{N_{ij}}$. Specifically,

$$e_k |_{e_i, e_j, e_{N_{ij}}} \sim \mathcal{N} \left(\frac{\rho}{1 + (d_{ij} + 1)\rho} \left(e_i + e_j + \sum_{k \in N_{ij}} e_k \right), V_{d_{ij}+2} \right) \quad (9)$$

where $d_{ij} := \#(N_{ij})$ and $V_{d_{ij}+2}$ denotes the variance of e_k conditional on observing $(d_{ij} + 2)$ endowment realizations.¹⁶

We again postulate a linear transfer rule: $t_{ij} = \alpha_{ij}e_i - \alpha_{ji}e_j + \sum_{k \in N_{ij}} \beta_{ijk}e_k + \mu_{ij}$, and plug in the postulated form to obtain a system of verification equations. Again, we ignore the verification equations for the state-independent transfers μ , and defer the discussion of μ to Section 6.3. After some tedious algebraic transformations, we again arrive at a rather complicated system of linear equations in (α, β) that defines the condition for Pareto efficiency, namely system (23) (See Lemma 7 in Appendix A and B.1 for its explicit expression). However, instead of solving for this complicated system directly, we first present an innocuous simplification of it. Due to the possible existence of cycles and superfluous transfers along cycles, this system may in general admit multiple solutions. For example, given a complete triad ijk , we can make a superfluous transfer of a ϵ share of e_i from i to j , j to k and k to i by adding ϵ to α_{ij}, β_{jki} , and subtracting ϵ from α_{ik} . It can then be checked that this operation is indeed superfluous, in the sense that $(\alpha_{ij} + \epsilon, \beta_{jki} + \epsilon, \beta_{kji} - \epsilon, \alpha_{ik} - \epsilon)$, keeping everything else fixed, still solves the system of equations for Pareto efficiency with the induced final consumption plan left unchanged. Since any amount of superfluous cycles are redundant, we can set $\beta_{ijk} = 0$ for all triads ijk without loss of Pareto efficiency. Hence, in the following, we establish that there exists some vector of strictly bilateral transfer shares $(\alpha^*, \beta^* \equiv \mathbf{0})$ that solves (23) and thus achieves Pareto efficiency. In other words, the strictly bilateral linear transfer rules that we characterize below are the “simplest” Pareto efficient rules in terms of minimizing the sum of state-contingent transfers.

By setting $\beta = \mathbf{0}$, we achieve a significant simplification of (23) and obtain the

¹⁶See, for example, Eaton (2007), p116-117.

following system:

$$\begin{cases} \alpha_{ij} = \frac{1}{2} \left(1 - \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} + \gamma_{ij} \right) & (10.1) \\ 0 = \alpha_{ki} - \alpha_{kj} + \gamma_{ij} \quad \forall k \in N_{ij} & (10.2) \forall i, j \text{ s.t. } G_{ij} = 1 \quad (10) \\ \gamma_{ij} = \frac{\rho}{1+(d_{ij}+1)\rho} \left(\sum_{k \in N_i \setminus \bar{N}_j} \alpha_{ki} - \sum_{k \in N_j \setminus \bar{N}_i} \alpha_{kj} \right) & (10.3) \end{cases}$$

The first equation (10.1) states that the share of e_i transferred from i to j is half of the remaining share after i 's transfers to i 's other neighbors plus the informational adjustment term between ij . With $\gamma \equiv 0$, which is implied by $\rho = 0$, α will be simply reduced to the local equal sharing rule. The second equation (10.2) requires that the difference in the shares of e_k undertaken by i and j is equal to the informational effect between ij , so that it is indeed optimal for ij to set $\beta_{ijk} = 0$. This confirms again that strict bilaterality ($\beta = \mathbf{0}$) is not an assumption, as (10.2) also incorporates the efficiency requirements for $\beta = \mathbf{0}$. The third equation (10.3) defines the auxiliary variable γ_{ij} . We interpret γ_{ij} as the net informational effect because it is the rate at which locally observed endowment realizations affect the pair ij 's joint expectation of non-local endowments. Notice that γ_{ij} is the same across $k \in \bar{N}_{ij}$ because each element of $(e_k)_{k \in \bar{N}_{ij}}$ provides exactly the same amount of information to the linked pair ij for their joint inference on non-local endowments. Given α , $|\gamma_{ij}|$ is decreasing in d_{ij} , indicating that the magnitude of the informational effect (for any single endowment realization) is decreasing in the amount of local information. Below we proceed to show the existence and provide a closed-form characterization of a solution to (10).

We first prove that (10.2) are implied by (10.1) and (10.3). By differencing (10.1) for ki and for kj we get: $\alpha_{ki} - \alpha_{kj} = \gamma_{ki} - \gamma_{kj}$. Hence, in the presence of (10.1) equation (10.2) is equivalent to, for all triads ijk , $\gamma_{ij} + \gamma_{jk} + \gamma_{ki} = 0$. This is reminiscent of the Kirchhoff Voltage Law for electric resistor networks, which states that the sum of voltage differences across any closed cycle must sum to zero. It turns out that the Kirchhoff Voltage Law indeed holds in our setting for any cycle in a general network.

Proposition 4. “Kirchhoff Voltage Law”: $\forall \rho \in \left(-\frac{1}{n-1}, 1\right)$, if (10.1) and (10.3) admit a unique solution (α, γ) , this solution also satisfy (10.2); furthermore, given any cycle $i_1 i_2 \dots i_m i_1$, γ satisfies the “Kirchhoff Voltage Law” $\gamma_{i_1 i_2} + \gamma_{i_2 i_3} + \dots + \gamma_{i_m i_1} = 0$.

Intuitively, Pareto optimality requires that ij share equally the net difference in the

conditional expectations of *nonlocal* inflow exposures (captured by γ_{ij}) by creating an opposite net difference in their *local* inflow exposures, as specified in equation (10.2). This adjustment guarantees the expectational Borch rule in equation (5), and therefore Pareto efficiency. To see this, notice that conditional expectation and variance of consumption will differ only by a constant across different local states (I_{ij}). Together, this implies that conditional CE's differ only by a constant, as required.

Given the redundancy of (10.2) in the presence of (10.1) and (10.3), we may now conclude that any solution to the system consisting of (10.1) and (10.3) defines a linear and Pareto efficient profile of transfer rules in \mathcal{T} . However, the matrix defined by (10) is difficult to work with, so we switch to an equivalent formulation of the system that is more tractable.¹⁷ Specifically, we establish next that the Pareto efficiency is equivalent to minimization of the sum of consumption variances among linear risk-sharing arrangements in \mathcal{T} , and then show that the Pareto efficient risk-sharing arrangement is closely related to a network statistic that aggregates all *even-length paths* for every household, weighted in a particular way.

Let α be a linear profile of transfer rules in \mathcal{T} , and consider the following optimization problem that minimizes the sum of each individual's consumption variance under the risk-sharing arrangements defined by α :

$$\min_{\alpha} \sum_{i \in N} Var \left[\left(1 - \sum_{j \in N_i} \alpha_{ij} \right) e_i + \sum_{j \in N_i} \alpha_{ji} e_j \right]. \quad (11)$$

Writing $\alpha_{ii} := 1 - \sum_{j \in N_i} \alpha_{ij}$, the minimization problem (11) is equivalent to

$$\min_{(\alpha_{ij}), (\alpha_{ii})} \sum_i \left(\sum_{j \in \bar{N}_i} \alpha_{ji}^2 + 2\rho \sum_{j, k \in \bar{N}_i, j < k} \alpha_{ji} \alpha_{ki} \right) \text{ s.t. } \sum_{j \in \bar{N}_i} \alpha_{ij} = 1 \quad \forall i \in N.$$

Let $\tilde{\Lambda}_i$ be the Lagrange multiplier associated with i 's outflow constraint $\sum_{j \in \bar{N}_i} \alpha_{ij} = 1$ and denote $\Lambda_i := \frac{\tilde{\Lambda}_i}{2(1-\rho)}$. It is then straightforward to check that the set of admissible shares α is convex and the objective function is convex in α (as the underlying variance-covariance matrix is positive definite). Then, taking the FOC for the La-

¹⁷Given Proposition 4, from now on, by (10) we mean the system of linear equations defined by (10.1) and (10.3).

grangian, we have

$$\begin{cases} \alpha_{ji} = \Lambda_j - \frac{\rho}{1-\rho} (\alpha_{ii} + \sum_{k \in N_i} \alpha_{ki}) & \forall j \in \bar{N}_i, \forall i \in N & (12.1) \\ \sum_{j \in \bar{N}_i} \alpha_{ij} = 1 & \forall i \in N & (12.2) \end{cases} \quad (12)$$

This is a system of $(\sum_i d_i + 2n)$ equations in $(\sum_i d_i + 2n)$ variables (α, Λ) .

In order to obtain a closed-form characterization of Pareto efficient linear transfer arrangements, we first show that if (12) admits a unique solution, then this unique solution also solves (10). This confirms that the solution to (12) indeed defines a Pareto efficient profile of transfer rules in \mathcal{T}^* . We then show that (12) admits a closed-form solution that characterizes the linear profile of transfers. In Section 4.3 we relate both of these solutions by defining two related centrality measures that can be used interchangeably to describe efficient transfers in our model.

Proposition 5. *$\forall \rho \in (-\frac{1}{n-1}, 1)$, if system (12) admits a unique solution, then the solution also solves system (10): i.e., a profile of linear and strictly bilateral transfer rules is Pareto efficient in \mathcal{T} if it uniquely minimizes the sum of consumption variances among all profiles of linear and strictly bilateral transfer rules in \mathcal{T}^* .*

We now show that, for any given network, system (12) indeed admits a unique solution that can be expressed in closed form. The solution depends on the pairwise correlation ρ and on the positions of individuals in the network, and can be represented as a linear function of accumulated paths along the network.

Proposition 6. ¹⁸ *For any $\rho \in (-\frac{1}{n-1}, 1)$ and any network structure G , or for $\rho = -\frac{1}{n-1}$ and any G such that $\max_{i \in N} d_i < n - 1$, there exists a unique solution to system (12) given by the following: $\forall i \in N, \forall j \in \bar{N}_i$,*

$$\alpha_{ji} = \Lambda_j - \frac{\rho}{1 + \rho d_i} \sum_{k \in \bar{N}_i} \Lambda_k \quad (13)$$

where Λ_i is given by:

¹⁸See Appendix B.3 for Pareto efficient risk-sharing arrangements in the boundary cases of $\rho \in \{-\frac{1}{n-1}, 1\}$.

- (Fixed point representation):

$$\Lambda_i = \frac{1}{d_i + 1} \left(1 + \sum_{j \in \bar{N}_i} \sum_{k \in \bar{N}_j} \frac{\rho}{1 + \rho d_j} \Lambda_k \right) \quad (14)$$

- (Closed-form representation): writing $\Lambda = (\Lambda_i)_{i=1}^n$

$$\Lambda = (\bar{D} - \bar{G}\Psi\bar{G})^{-1} \mathbf{1}$$

where \bar{D} is a diagonal matrix with its i -th diagonal entry being $d_i + 1$, Ψ is a diagonal matrix with its i -th diagonal entry being $\frac{\rho}{1 + \rho d_i}$, and $\bar{G} := G + \mathbf{I}_n$.

- (Explicit representation): For $\rho \in [0, 1)$,

$$\Lambda_i = \frac{1}{d_i + 1} + \sum_{q \in \mathbb{N}} \sum_{j \in N} \sum_{\pi_{ij} \in \Pi_{ij}^{2q}} W(\pi_{ij}) \quad (15)$$

where $W(\pi_{ij})$, the weight of each path $\pi_{ij} = (i_0, i_1, i_2, \dots, i_q)$ of length q from i to j (i.e. $i_0 = i$ and $i_q = j$), is given by,

$$W(\pi_{ij}) := \frac{1}{d_{i_0} + 1} \cdot \frac{\rho}{1 + \rho d_{i_1}} \cdot \frac{1}{d_{i_2} + 1} \cdot \frac{\rho}{1 + \rho d_{i_3}} \cdot \dots \cdot \frac{1}{d_{i_q} + 1} \quad (16)$$

In particular the proof shows that the system of equations (14) has a unique solution, characterized by the explicit representation. The representation reveals that the form in which the network determines the Pareto efficient linear transfer arrangements has to do with interaction at distance two (i.e. neighbors of neighbors).

Intuitively, the network interaction terms in (12.1) define substitutability across the shares going to j . This implies that individuals *at most two links apart* (i.e. with a common neighbor, i) affect each others' transfer shares directly. But indirect effects play a crucial role here as well. To see this, notice that these two households not only interact through their transfer to i , but might also exchange resources with other partners, and these other relations affect what i receives from them, given their constraints in (12.2). This is the main message behind equation (13), where these inter-dependencies along the network have been worked out, and we can express the share from j to i as a function of some constants Λ 's, that accumulate all these

indirect effects. These values can be shown to depend across individuals according to the recursive formulation in (14), which is reminiscent of Katz-Bonacich, Page Rank, and other global network measures, albeit with two crucial differences: 1) The centrality of i depends on the centralities not of direct neighbors, but of neighbors of neighbors (i.e. at length two), and 2) the weights are not a simple geometric series (as in the Bonacich measure), but instead depend explicitly on the degree of the direct neighbors that are linking i with all of her length-2 neighbors.

The recursive formulation in (14) admits a unique solution, for any network. This provides an alternative characterization of the centrality as the accumulation of *weighted even paths*, which exhibit the two crucial differences described above (i.e. length two and path-specific weights). However, notice that (14) sums over individuals in \bar{N}_i and \bar{N}_j . In other words, self-loops are allowed. This implies that we are not in a situation where an individual that is, say, at distance 3 from i will not matter for i 's centrality measure. On the contrary, she will in fact matter because self-loops will allow us to reach any individual that is weakly connected to i . However, the weighting scheme crucially depends on the even-length paths that we can compute, starting from i . In other words, while this measure ultimately relates individuals at all distances in the network, the specific weights between each pair of individuals require accounting only the even-length paths that connect them.

This complicated weighting scheme unfortunately makes comparative statics on the network difficult to analyze. To see this notice that when a link is removed from the network a number of even-length paths disappear, lowering the total elements summing in (15). However, this also lowers the degree (or connectivity) of the two individuals involved in that link. This increases the weights associated to all even-length paths that go through each of these two individuals, as shown in equation (16). As such, it is difficult in general to know which way the centrality measure moves as links are deleted. In any case, it is often helpful to think of the general result in Proposition 6 as capturing the extent to which the sender's indirect interactions in the network cannot be accessed by any other of the receiver's partners.

Having characterized Pareto efficient risk-sharing arrangements subject to the local information constraint with globally correlated endowments, we now proceed to describe some interesting properties that follow from these results.

4.3 Properties of Efficient Risk Sharing with Local Information

4.3.1 Network Centrality and Efficient Transfers

Proposition 6 and Proposition 4 suggest two potential measures of network centrality that, as functions of the network structure G and the correlation parameter ρ , directly enter into the determination of the Pareto efficient transfer shares. In the following section we explicitly define these two centrality measures, provide interpretations for them, and examine the relationship between them.

We first consider a centrality measure suggested by Proposition 6. In observation of the “Kirchhoff Voltage Law”, $\forall \rho \neq 0$, there exists a function $V : N \rightarrow \mathbb{R}$ s.t.

$$V(i) - V(j) = \gamma_{ij} \equiv \frac{\rho}{1 + (d_{ij} + 1)\rho} \left(\sum_{k \in N_i \setminus \bar{N}_j} \alpha_{ki} - \sum_{k \in N_j \setminus \bar{N}_i} \alpha_{kj} \right).$$

In particular, V can be constructed from γ in the following way. Fix any individual, say, individual 1, and normalize $V(1) = 0$. For any other individual $j \neq 1$, as the network is connected, there exists a path $1i_1 \dots i_m j$ that connects 1 to j . Then we simply define $V(j) = \gamma_{1i_1} + \dots + \gamma_{i_m j}$. By Proposition 4, V is well-defined.

Using terminology of electrical resistor networks, $V(i)$ is analogous to an “energy potential” for node i , and γ_{ij} is the “potential difference” or “voltage” between i and j . Given γ , the “current flows” α (or more precisely, its deviation from local equal sharing) are driven by γ according to (10.1). However, γ is simultaneously determined by the currents α according to (10.3). The multiplier $\frac{\rho}{1+(d_{ij}+1)\rho}$ in the Γ matrix captures how much a pair ij “discounts” nonlocal inflows given the amount $(d_{ij} + 2)$ and the quality (ρ) of local information. Despite the possible differences in this discount multiplier across linked pairs, the optimum is associated with a globally consistent assessment of each individual’s “net position in nonlocal risk exposures”, summarized by $V(i)$. We refer to $V(i)$ (or simply V_i) as i ’s “*in-potential centrality*”.

The other centrality measure we consider is the Lagrange multiplier Λ_i in Proposition 6, which is a network statistic that aggregates all even-length paths for every household, weighted in some particular way given by (16). In contrast with the in-potential centrality V_i , which is based on net difference in inflow shares, Λ_i relates to net difference in outflow shares. Hence, we refer to Λ_i as the “*out-potential centrality*”.

Notice that recursive expressions in the flavor of (14) are often found in the definition of centrality measures in network analysis. For instance, in their well-known work on strategic complementarity in networks, Ballester, Calvó-Armengol, and Zenou (2006) show that equilibrium actions depend on a similar recursive measure known as Bonacich Centrality. More recently, Banerjee, Chandrasekhar, Duflo, and Jackson (2014) have sought to identify individuals in the network that are best placed to diffuse information on micro-credit opportunities in India. They find that participation is higher if those first informed have higher eigenvector centrality.¹⁹ Moreover, a recent randomized control trial in Malawi by Beaman, Ben Yishay, Magruder, and Mobarak (2015) has shown that network theory-based targeting strategies such as these can increase technology adoption in village economies by up to 3 percentage points, relative to assignment by extension workers.

All of these measures can be expressed generically as

$$B_i = c + \gamma \sum_k g_{ik} B_k \quad (17)$$

for some constant c and with $|\gamma| < 1$. This expression essentially says that i 's measure depends linearly on the sum of measures that are connected to i . However, there are two crucial distinguishing differences with respect to the out-potential centrality measure defined in (15). First of all, notice that equation (14) does not sum over all centrality measures of i 's neighbors, but instead sums over those of i 's neighbors' neighbors. In other words, the out-potential centrality is defined recursively at distance two, not one. This is not unique in network analysis. It appears in some work on vertex similarity by Jeh and Widom (2002), and it has also appeared in newer page-ranking algorithms, such as the HITS algorithm.²⁰ Secondly, notice that, in contrast to equation (17), the out-potential centrality does not weight all measures at distance two with a common parameter γ . Instead, they are weighted by the degree of the household that serves as a bridge between them. So for example, imagine two households k and i are both linked to a third household l , but are not linked to

¹⁹Already at the beginning of the Internet boom, a number of algorithms surfaced that allowed users to rank websites by their significance in the broader world wide web network. Procedures such as PageRank and HITS algorithm also refined measures recursively throughout the network.

²⁰In the HITS algorithm, a webpage is given both an authority and a hubness score, with the property that a website's authority is determined by the sum of the hubness scores of other websites it links to, while a website's hubness is determined by the sum of the authorities of websites it is linked by. This implies that each one of these measures is defined recursively at distance two.

each other. Then, k 's measure will enter the definition of i 's measure, weighed by the degree of l . The type of weighting scheme in equation (16) can be thought of in terms of the accumulated local interactions. Recall that indirect interactions only represent the concatenation of various direct interactions linked together by the network constraints, and the weights $\frac{\rho}{1+\rho d_k}$ capture all the households in a given path engaged in direct interactions and the weights $\frac{1}{d_k+1}$ capture the connecting household's constraint. This weighting scheme marks a crucial distinction vis-à-vis other measures, in that additional paths does not guarantee an increase in households' network measure.

We now relate the two centrality measures defined above to the Pareto efficient transfer shares in the next proposition, which also help explain the terms “in/out-potential centralities”.

Proposition 7. *Let α be a strictly bilateral Pareto efficient transfer shares. Write $\Lambda_{ij} := \Lambda_i - \Lambda_j$. Then, for any linked pair ij and any $k \in \bar{N}_{ij}$,*

$$(i) \quad \gamma_{ij} = \alpha_{kj} - \alpha_{ki}.$$

$$(ii) \quad \Lambda_{ij} = \alpha_{ik} - \alpha_{jk}.$$

$$(iii) \quad \Lambda_{ij} - \gamma_{ij} = \alpha_{ii} - \alpha_{jj}.$$

By construction, γ_{ij} is based on net differences in inflow shares while Λ_{ij} is based on net differences in outflow shares. If i has a larger potential centrality than j ($\gamma_{ij} > 0$), i gets larger net nonlocal exposures (from his noncommon neighbors $N_i \setminus \bar{N}_j$) than j does, and at Pareto optimum this net difference is shared between i and j through an opposite difference in net exposures to common friends: $\alpha_{ki} - \alpha_{kj} = -\gamma_{ij} < 0$, i.e. i typically gets a smaller local exposure (from a common neighbors \bar{N}_{ij}) than j does. A corresponding relationship also holds for the out-potential centrality. If i has a larger out-potential centrality than j ($\Lambda_{ij} > 0$), i gives out a larger share of his own shock locally (to a common neighbor in \bar{N}_{ij}) than j does: $\alpha_{ik} - \alpha_{jk} = \Lambda_{ij} > 0$ for $k \in \bar{N}_{ij}$, and therefore gives out a smaller share of his own shock nonlocally than j does. Also, the discrepancy between the potential centrality difference γ_{ij} and the out-potential centrality difference Λ_{ij} equals the difference in self-shock exposures $\alpha_{ii} - \alpha_{jj}$.

4.3.2 Network Centrality and Consumption Variance

We seek to establish a relationship between network centrality of an individual and her consumption variance, given a Pareto efficient risk-sharing arrangement. The

fact that, in a Pareto optimal arrangement, neighbors equally share expected state-dependent realizations conditional on their local state does not imply that their ex ante consumption variance is equal. The transfer scheme that achieves the equalization of conditional expectations of the state-dependent part of the consumption plan has the feature that the neighbor with a higher exposure to non-common endowment shocks ends up with higher consumption variance. Mathematically this is because conditional expectations of uncommon shocks are equal to a constant $\frac{\rho}{1+(d_{ij}+1)\rho}$ times the sum of common shocks, and this constant is strictly smaller than 1. Hence, a unit increase in exposure to non-common shocks is compensated by less than a unit decrease in exposure to common shocks. This implies that individuals who are more central in risk sharing end up with a higher consumption variance in our model.

We illustrate the above relationships between network centrality, transfer flows, and consumption variances in the context of star networks. Let c denote the center individual, who is connected to $n-1$ peripheral individuals, and none of the peripheral individuals are connected to each other. We use p to refer to a generic peripheral individual.

It is straightforward to show that a linear risk-sharing arrangement achieving Pareto efficiency subject to local information constraints specifies the following endowment shares to be transferred:

$$\alpha_{cp} = \frac{2 + 2(n-1)\rho}{n(2+n\rho)}, \quad \alpha_{pc} = \frac{1+\rho}{2+n\rho}, \quad \gamma_{cp} = \frac{(n-2)\rho}{2+n\rho}.$$

Given this risk-sharing arrangement, if the potential centrality of peripheral individuals is normalized to 0, the potential centrality of the center is $V(c) = \frac{n-2}{2+n\rho}$. It can be shown that the difference in consumption variances in efficient contracts satisfies

$$\text{Var}(x_c) - \text{Var}(x_p) = \frac{(n-2)(1+(n-1)\rho)(1-\rho^2)}{(2+n\rho)^2} \geq 0$$

with equality only at $\rho \in \{-\frac{1}{n-1}, 1\}$. In particular, $\text{Var}(x_c) - \text{Var}(x_p) \rightarrow \frac{1-\rho^2}{\rho}$ as $n \rightarrow \infty$, and hence the consumption variance of the center can be much higher than the consumption variance of a periphery individual when ρ is low and n is high.

Centrality in risk sharing, as implied via the Pareto efficient risk-sharing arrangements subject to local information constraints, is not equivalent to standard notions of centrality, such as degree or eigenvector centrality. However, on typical networks

they are highly positively correlated, suggesting that our model predicts a positive relationship between these centrality measures and consumption variance. This contrasts with the predictions of the model in AMS, in which enforcement constraints limit the efficiency of risk-sharing arrangements. We illustrate this point numerically, estimating the correlation predicted by our model and by the model in AMS, between an individual’s centrality and consumption variance, via simulated endowment realizations in two real-world village networks from India from two different databases, each randomly selected and provided us by the researchers who collected the data.²¹

In both simulations, we randomly drew the endowment $e_i^{(t)}$ of each household according to the standard normal distribution for $T = 5000$ times: $\{e_i^{(t)}\}_{i,t} \sim iid \mathcal{N}(0, 1)$. We assumed that all households have CARA utility functions with $\lambda = 1$. We then computed the final consumptions of each household under the equally-weighted Utilitarian optimal risk-sharing arrangement subject to local information constraints, using the results from subsection 4.1, and the sample variance of final consumptions for each household (note that the variance does not depend on the planner’s weights). Following this we computed the sample correlation between degree/eigenvector centrality and consumption variance. Similarly, we computed the constrained efficient consumptions implied by the model in AMS, and the sample correlation between the centrality measures and consumption variance under three levels of capacity constraints (the maximum amount that can be transferred through any link, at any state): 0.5, 1 and 1.5. The results are summarized in Table 1. All results are highly statistically significant with p-values very close to zero, except for the correlation between consumption variance and eigenvalue centrality for one of the two datasets).²²

Under the AMS model, we observe a negative correlation between centrality and

²¹The first network was provided to us by Erica Field and Rohini Pande, who collected it from villages in the districts of Thanjavur, Thiruvarur and Pudukkotai (Tamil Nadu) in India. In a subset of the villages, complete within-village network data was collected by surveying all households. The second network is from data collected by Abhijit Banerjee, Arun Chandrasekhar, Esther Duflo and Matthew Jackson in Karnataka, India (they collected complete within-village network data in 75 villages), used for example in the [Banerjee, Chandrasekhar, Duflo, and Jackson \(2014\)](#). From both datasets we received the network of financial connection for one randomly selected village with complete network data. From the original network we created the undirected “AND” network, that is, we defined a link between two households whenever both households indicated each other as a borrowing relationship. We excluded households that became isolated in the “AND” network.

²²The p-values, calculated from standard t -tests against the null hypotheses of zero correlations, are at orders of magnitudes below 10^{-10} , except the case noted.

Table 1: Correlation between Centralities and Consumption Variances

		(A) Field & Pande		(B) BCDJ	
	Capacity	Degree	Eigenvector	Degree	Eigenvector
AMS	0.5	-0.8943***	-0.4585***	-0.3552***	-0.3155***
	1.0	-0.6885***	-0.2313***	-0.2913***	-0.2573***
	1.5	-0.5430***	-0.0898***	-0.2843***	-0.2511***
Our Model		0.1994***	0.1659***	0.2084***	0.0858

*** denotes statistical significance at 1%-level.

consumption variance. In AMS, transfers along links are subject to capacity constraints. As a result, centrally located households tend to have a lower consumption variance, because capacity constraints are less likely to be binding for them locally, and for typical endowment realizations they end up pooling risk with a larger set of other households.²³ This holds for all capacity values we used in the simulations, but the relationship is more highlighted for relatively stricter capacity constraints.²⁴

Under the model of the current paper, we observe the opposite sign: sample correlation between both degree and eigenvector centrality on the one hand, and consumption variance on the other hand is positive (as noted above, not significantly for eigenvalue centrality when using the BCDJ data).

5 Implications for Empirical Tests of Risk Sharing

The performance of risk-sharing communities has been repeatedly tested in data since the work of [Cochrane \(1991\)](#), [Mace \(1991\)](#) and [Townsend \(1994\)](#). Their original approach developed empirical tests of full insurance that related household consumption and income. Indeed, the well known Borch rule – equating the ratio of marginal utilities across households – imposes that, under full insurance, household consumption should not respond to idiosyncratic movements in income after controlling for aggregate shocks. This implication can be tested in the following popular regression:

$$\log(c_{it}) = \alpha_i + \beta_1 \log(y_{it}) + \beta_2 \log(\bar{y}_t) + \epsilon_{it} \quad (18)$$

²³Using terminology from AMS, more centrally-located households typically end up on larger “risk-sharing islands.”

²⁴As capacities increase, centrality in the AMS model matters less, since capacity constraints are less likely to bind.

where c_{it} and y_{it} correspond to household i 's consumption and income at time t , and where $\bar{y}_t = \sum_i y_{it}$ represents aggregate village income at time t .²⁵ Full insurance implies that $\beta_1 = 0$ and $\beta_2 = 1$. An overwhelming proportion of studies have rejected the full-insurance hypothesis in a wide number of settings. As a result, a great deal of work has followed, that seeks to explain this stylized fact.

On the theory side, we have argued that this paper complements an ongoing effort to model the relevant contracting frictions in informal risk sharing environments.²⁶ In this section we argue that our framework also responds to a recent strand of the literature that suggests modifying the classical Townsend test in order to accommodate various forms of heterogeneity. Some of this work argues that the standard consumption regression in (18) is misspecified if, for instance, households hold heterogeneous risk preferences.²⁷ More relevant to the current discussion, several other studies have also suggested that households within a village indeed access different risk sharing groups, and that controlling for aggregate-level shocks, as in (18), would incorrectly estimate income coefficients: \bar{y} should be group-specific. In a couple well-known examples, [Mazzocco and Saini \(2012\)](#) argue that the relevant sharing group in India is the caste (rather than the village), while [Attanasio, Meghir, and Mommaerts \(2015\)](#) test for efficient insurance within extended families in the U.S.²⁸

This paper refines and generalizes the modified tests that evaluate the performance of insurance mechanisms on local sharing groups. Rather than taking groups as separate, perfectly insured communities, the current framework allows for a fully general social structure with interconnected sharing groups that are specific to each household, and which may overlap in complicated ways along any given network. We show how, under the local information constraints of our model, not defining the relevant local sharing group biases the estimates of risk-sharing tests. More importantly, we show that controlling for this bias will not eliminate the correlation between household consumption and income: the structure of the network, coupled with the

²⁵Village-time fixed effects are traditionally used to capture aggregate shocks at the village level.

²⁶For example [Thomas and Worrall \(1990\)](#), [Kocherlakota \(1996\)](#), [Ambrus, Mobius, and Szeidl \(2014\)](#), and [Kinnan \(2011\)](#).

²⁷See for instance [Mazzocco and Saini \(2012\)](#) and [Schulhofer-Wohl \(2011\)](#).

²⁸In similar procedures [Hayashi, Altonji, and Kotlikoff \(1996\)](#) consider whether extended families can be viewed as collective units sharing risk efficiently. [Munshi and Rosenzweig \(2016\)](#) also find that the caste is the relevant group to explain migration patterns in rural India. Most relevant here, [Fafchamps and Lund \(2003\)](#) address the failure of efficient insurance in the data suggesting that households receive transfers not at the village level, but from a network of family and friends

information constraints, induces imperfect risk-sharing and generates heterogeneity in sharing behavior. The current framework therefore allows us to decompose the standard Townsend coefficient β_1 into an underlying distribution of household-specific coefficients that capture the varying risk-sharing possibilities induced by the network structure, and which can be interpreted economically in terms of consumption volatility (as shown in the previous section).

To fix ideas, consider the simple network with three individuals and independent endowments in section 2 and set $\lambda_i = 1$; all arguments below can be extended to general networks, correlated endowments, and any profile of Pareto weights. If we write down final consumption for each household in the form of the classical risk-sharing specification of equation (18), we have that,

$$\begin{cases} c_{1t} = \alpha_1 + \left(\frac{1}{3} - \frac{1}{2}\right) y_{1t} + \frac{1}{2}\bar{y}_t + \epsilon_{1t}, \\ c_{2t} = \alpha_2 + \left(\frac{1}{2} - \frac{1}{3}\right) y_{2t} + \frac{1}{3}\bar{y}_t + \left(\epsilon_{2t} - \frac{1}{3}y_{3t}\right), \\ c_{3t} = \alpha_3 + \left(\frac{1}{2} - \frac{1}{3}\right) y_{3t} + \frac{1}{3}\bar{y}_t + \left(\epsilon_{3t} - \frac{1}{3}y_{2t}\right), \end{cases}$$

where $\alpha_1 = \frac{1}{12}r\sigma^2$ and $\alpha_2 = \alpha_3 = \frac{1}{24}r\sigma^2$ correspond to state-independent transfers and are represented as household-specific intercepts. These equations reflect three important themes of this paper as they relate to empirical tests of risk-sharing: 1) coefficients on own income are generically different from zero for all households. i.e. $\alpha_{ii} \neq \alpha_{ij}$, 2) these coefficients vary according to households' network position, and 3) imposing the common sharing group on all households generates biased estimates: notice the last two equations contain weighted incomes in the error term. The classical risk sharing test in (18) pools these equations and obtains a unique estimate for β_1 ; given the previous discussion we expect this estimate to be biased, different from zero, and positive.

In order to obtain unbiased estimates for β_1 , consider estimating (18) with the relevant local sharing group instead. In this case, we show coefficients are properly estimated, but we still obtain heterogeneous estimates, β_i , for the coefficients on own income. As a result, the risk sharing test still delivers positive estimates – not surprisingly, since risk sharing is not efficient under information constraints. To see this, rewrite again our consumption equations in the form of (18), but now allow for household-specific aggregates, $\bar{y}_{it} = \sum_{j \in N_i} y_{jt}$, that sum over the incomes of i 's sharing partners. In this case we have,

Table 2: Simulated Risk-Sharing Test under the Model for Two Simple Economies

	<i>Dependent variable: Consumption</i>			
	<i>Star Network</i>		<i>Circle Network</i>	
	Common Group (1)	Local Group (2)	Common Group (3)	Local Group (4)
Income	0.201 (0.002)	0.027 (0.002)	0.121 (0.001)	0.001 (0.001)
Agg. Income	0.780 (0.001)	0.977 (0.002)	0.845 (0.002)	0.998 (0.001)
Observations	300,000	300,000	400,000	400,000
R ²	0.529	0.692	0.477	0.654

Note: Income data simulated from log-normal distribution with $\sigma^2 = 4$ and $t = 100,000$. Model estimated on logged data with household-specific intercepts. Values in parentheses are standard errors.

$$\begin{cases} c_{1t} = \alpha_1 + \left(\frac{1}{3} - \frac{1}{2}\right) y_{1t} + \frac{1}{2} \bar{y}_{1t} + \epsilon_{1t}, \\ c_{2t} = \alpha_2 + \left(\frac{1}{2} - \frac{1}{3}\right) y_{2t} + \frac{1}{3} \bar{y}_{2t} + \epsilon_{2t}, \\ c_{3t} = \alpha_3 + \left(\frac{1}{2} - \frac{1}{3}\right) y_{3t} + \frac{1}{3} \bar{y}_{3t} + \epsilon_{3t}, \end{cases}$$

Because aggregate income terms are now household-specific (i.e. \bar{y}_i), the additional terms in the error disappear and we obtain unbiased estimators. Notice, however, that coefficients to own income are different from zero so long as $\alpha_{ii} \neq \alpha_{ij}$. This implies that the pooled regression will again deliver positive coefficient for β_1 , even with the appropriate local aggregates. In this context, the pooled estimate in fact represents the average of the underlying heterogeneity in risk-sharing possibilities across households, which respond to network effects and relate to consumption volatility as specified by the theoretical results above.

Finally, notice that under sufficiently symmetric structures, we cannot reject this localized version of the Townsend test, because in “regular” networks $\alpha_{ii} - \alpha_{ij} = 0$. This means we are able to generalize the discussion on appropriate local aggregates in Townsend regressions – the theory is sufficiently rich to accommodate previous models of local sharing groups, as well as many other local structures. In fact, a well-defined local version of the Townsend test may fail to reject full insurance not only if castes or extended families are perfectly connected partitions (as stressed in the previous literature), but also if the social structure is sufficiently symmetric. As an

extreme example, consider the circle network in which all individuals are identically positioned. Although all local sharing groups overlap and none of them are perfectly connected, this network structure would nonetheless generate sufficient regularity to “pass” an appropriately defined version of the risk-sharing test.

The previous discussion can be observed compactly in table 2, where the risk-sharing test is performed on simulated income data for the three individual “star” network discussed above, and the four individual “circle” network that exhibits perfect symmetry. The test is performed both with a common aggregate income term (columns 1 and 3) and with appropriately defined local sharing groups (columns 2 and 4). Notice that coefficients on own income are biased upwards by a whole order of magnitude when imposing a common aggregate income term but remain positive and significant in the star network, where the lack of symmetry keeps the pooled coefficient estimate away from zero. However, as discussed above, the circle network “passes” the Townsend test (coefficient to income is not significant) under appropriately specified local aggregate income terms.

6 Discussion

6.1 Spatial Correlation Structure

In the previous section we considered a symmetric correlation structure, in which the correlation between the endowments of two individuals did not depend on their positions on the network. An alternative specification, however, is to incorporate the possibility of spatially correlated endowments, that is correlation that decays with social distance.²⁹ As we illustrate below (and in more details in Appendix B.4), this type of correlation structure can be detrimental to the efficiency of informal risk sharing with local information constraints.

For concreteness we assume that the correlation between e_i and e_j geometrically decays with the social distance between i and j : $Corr(e_i, e_j) = \rho^{dist(i,j)}$, where the social distance $dist(i, j)$ is formally defined as the length (i.e., the number of links) of the shortest path connecting i and j in network G . Also, for analytical simplicity we focus on circle networks with $n = 2m + 1$ individuals. In order to make comparable

²⁹There are many reasons why this correlation structure is more realistic for certain types of endowment shocks: for example, as shown in Fafchamps and Gubert (2007) and in Conley and Udry (2010), social distance tends to be highly correlated with geographic proximity.

the risk-sharing efficiencies under geometrically decaying spatial correlation structure with that under the uniform global correlation structure analyzed in Section 4, we control the “shareable risk” to be the same across the two specifications by setting $\rho = \rho_m(\varrho) := \frac{\varrho(1-\varrho^m)}{m(1-\varrho)}$, where ρ is the uniform global pairwise correlation, while ϱ is the rate of decay in the geometrically decaying correlation structure. Then informal risk sharing subject to the local information constraint achieves drastically different levels of asymptotic efficiency under the two correlation structures.

Proposition 8. *Let $x_i^{unif}(\rho)$, $x_i^{geo}(\varrho)$ denote the Pareto efficient consumption plan subject to the local information constraint under the uniform and the geometrically decaying correlation structures, parametrized by ρ and ϱ respectively, and let $Var_{unif,\rho}$, $Var_{geo,\rho}$ correspond to the variance operators under the two probability distributions induced by the two correlation structures. Then:*

$$\lim_{\varrho \rightarrow 1} \lim_{m \rightarrow \infty} Var_{unif,\rho_m(\varrho)} \left(x_i^{unif}(\rho_m(\varrho)) \right) = \frac{1}{3},$$

$$\lim_{\varrho \rightarrow 1} \lim_{m \rightarrow \infty} Var_{geo,\rho} (x_i^{geo}(\varrho)) = 1.$$

Hence, for ϱ close to 1 and sufficiently large m , uniform correlation leads to significant risk sharing (yielding payoffs close to that under independent endowments), while geometrically decaying correlation yields payoffs very close to the autarky payoffs, even though the two correlation structures lead to the same payoffs if global information can be used for risk sharing.

This difference in risk-sharing efficiency, driven by the difference in underlying correlation structures, is a peculiar feature of the local information constraint considered in this paper. With global information, a geometrically decaying correlation structure does not in itself imply risk-sharing inefficiency relative to the uniform correlation structure. For example, in a large ring network considered above, shocks that are spatially far away from each other are almost independent, and each given individual is spatially far away from most of the individuals in the network. Hence, under global information mostly shocks with low correlations are pooled together, thus yielding significant risk reduction. However, with local information, only spatially close shocks are pooled, rendering risk sharing virtually ineffective due to the high local correlation.

This might help explain why it is the case that while in most settings empirical

research found that informal insurance works well, [Kazianga and Udry \(2006\)](#) found a setting in which informal insurance does not seem to help, and [Goldstein, de Janvry, and Sadoulet \(2001\)](#) found that certain types of endowment shocks are not well insured through informal risk sharing. In particular, this may be due to high correlation between endowments of neighboring households in the above settings, for the types of endowment shocks investigated.

6.2 Alternative Model Specifications

The main results in Section 4 are developed under the CARA-normal setting (Assumption 1) with a global correlation structure. We now consider the extendability of those results under some alternative model specifications.

Quadratic Utility Function

As to the specification of utility functions, we could alternatively work with quadratic utility functions, $u_i(x_i) = x_i - \frac{1}{2}rx_i^2$ for $i \in N$, which also admits a mean-variance expected utility representation. Noting that $u'_i(x_i) = 1 - rx_i$, the conditional Borch rule in Proposition 1 takes the form of $\lambda_i(1 - r\mathbb{E}_{ij}[x_i]) = \lambda_j(1 - r\mathbb{E}_{ij}[x_j])$. With equal Pareto weightings ($\lambda = \mathbf{1}$) and normal endowments, it can be shown that this leads to exactly the same system of linear equations as in (10).³⁰ Hence, the linear transfer shares given in Proposition 6 also characterize a Pareto efficient risk-sharing arrangement under the quadratic-normal setting. However, the Pareto efficient frontier traced out by all admissible Pareto weightings will correspond to a collection of different state-dependent transfer shares α .

Relaxing the Normality of the Endowments Distribution

The family of normal distributions have two properties that are technically essential to the proof of Pareto efficiency via the conditional Borch rule. First, a linear combination of a jointly normal vector remains normal, which allows us to explicitly characterize the distribution of final consumption $x_i = e_i - \sum_{j \in N_i} \alpha_{ji}e_j$ when transfer rules are linear. Second, normal distributions admit linear conditional expectations in the form of (9), which allows us to transform the conditions for Pareto efficiency into a system of linear equations on transfer shares. The assumption of normality

³⁰The proof is available in Appendix B.5.

can be relaxed slightly: if the endowment vector has a joint elliptical distribution,³¹ then both properties carry over,³² and thus the transfer shares given by Proposition 6 continue to characterize the Pareto efficient risk-sharing arrangements. Without the joint normality (or ellipticity) assumption, linear risk-sharing arrangements are in general not Pareto efficient. For example consider again the 3-individual line network with the random endowment vector $e = (Y, Z, -Z^3)$ where Y, Z are independent standard normal random variables. As there are effectively no uncertainty, the unique Pareto efficient profile of transfer rules is given by $t_{12}(e_1, e_2) = \frac{1}{3}e_1 - \frac{2}{3}e_2 - \frac{1}{3}e_2^3$, $t_{13}(e_1, e_3) = \frac{1}{3}e_1 - \frac{2}{3}e_3 - \frac{1}{3}e_3^{1/3}$ which are clearly nonlinear.

Heterogeneity in Expected Endowments

Throughout Section 4 we maintained the specification that endowment distributions have zero mean. However, we argue that, as risk sharing is the sole concern of this paper, the specification of zero mean is a warranted normalization. For concreteness, let \bar{y}_i be the expected level of endowment for individual i , and $y_i = \bar{y}_i + e_i$ be the random realization of endowment, where e_i is assumed to have zero mean. Clearly y and e induce the same local information structures $\sigma(y_k : k \in \bar{N}_{ij}) \equiv \sigma(e_k : k \in \bar{N}_{ij})$, so it makes no difference whether the risk-sharing arrangements are specified to be contingent on y or e . Hence our results remain valid regardless of whether “endowments” or “endowment shocks” are shared. Moreover, neither does it make any difference whether the linear “guess” is taken to be $t_{ij} = \alpha_{ij}e_i - \alpha_{ji}e_j + \sum_{k \in N_{ij}} \beta_{ijk}e_k + \mu_{ij}$ or $t_{ij} = \alpha_{ij}y_i - \alpha_{ji}y_j + \sum_{k \in N_{ij}} \beta_{ijk}y_k + \tilde{\mu}_{ij}$: both will lead to the same system of linear equations in (10), so the Pareto efficient state-dependent transfer shares are given by exactly the same formulas in Proposition 6, irrespective of the value of mean-income vector \bar{y} . Any difference induced by \bar{y} is completely absorbed by the state-independent transfers μ , which are irrelevant to Pareto efficiency in our framework.

Communication of Endowments of k th-order Neighbors

The specification of our model assumes that each individual can only observe endowment realizations of individuals at most one link away from him. Alternatively, we could assume local observability of endowment realizations at most k links away.

³¹Normal distribution is a special case of elliptical distribution.

³²See, for example, Fang, Kotz, and Ng (1990), Theorem 2.16 & 2.18.

For example, if we assumed that individuals could credibly communicate a neighbor’s endowment realizations to other neighbors, that would achieve shared knowledge of endowments $k = 2$ distance away.³³

Our model can be easily adapted to characterize the Pareto efficient risk-sharing arrangements under this relaxed observability assumption. Specifically, given a connected “physical network structure” G , we may define the “informational network structure” \tilde{G} by adding a link between every pair of individuals with no larger a graph distance than k in G . The Pareto efficient consumption plan is then induced by the state-dependent transfer shares given by Proposition 6 with \tilde{G} in place of G , as now \tilde{G} induces the appropriate local information structure for the analysis in Section 3 and 4.

6.3 State-Independent Transfers

In the previous sections we primarily focused on state-dependent transfer shares α , the only relevant variable under the CARA-normal setting as far as Pareto efficiency is concerned. We now provide a brief analysis of the state-independent constant transfers μ . Fixing any admissible Pareto weightings λ and any vector of expected endowments \bar{y} , there will be a profile of state-independent transfers μ^* associated with a solution to the the social planner’s problem. μ^* can be in general decomposed into three components: $\mu_{ij}^* = \hat{\mu}_{ij}(\sigma^2) + \bar{\mu}_{ij}(\bar{y}) + \tilde{\mu}_{ij}(\lambda)$, with $\hat{\mu}_{ij}(0) = 0$, $\bar{\mu}_{ij}(\bar{y}) = 0$ if $\bar{y} \in \text{Span}(\mathbf{1})$, and $\tilde{\mu}_{ij}(\lambda) = 0$ if $\lambda \in \text{Span}(\mathbf{1})$. The first term $\hat{\mu}_{ij}$ admits a clear economic interpretation as the “insurance premium”: individuals at different network positions undertake different exposures of endowment shocks and linked pairs typically exchange shocks of different sizes, so those who are subject to relatively larger final consumption variance are compensated through $\hat{\mu}_{ij}$. This component of constant transfers is of economic significance in real risk-sharing environments, as it relates to the division of risk-sharing surpluses and thus affect incentives in endogenous network formation. (See the next subsection for a more detailed analysis). The second term $\bar{\mu}$, however, should be interpreted as “inequality reduction”, as this term serves to equalizes consumption even if there is no risk to share at all ($\sigma^2 = 0$), and is thus irrelevant to the focus of this paper. Hence this term should be always set

³³Note however that cheap talk communication would not be credible in our context, as an individual would have an incentive to claim that noncommon neighbors had very bad endowment realizations, as that would increase the transfers she receives.

to zero if risk sharing is the sole concern, and thus the transfer arrangement can thus be interpreted as the sharing of “endowment shocks” e . This also partially motivated our focus on the case of $\bar{y} = \mathbf{0}$ in addition to the remark made in the last subsection. Lastly, the third term $\tilde{\mu}$ is driven by the arbitrariness in welfare weightings: throughout the paper this remains as a auxiliary variable mathematically and does not admit a clear economic interpretation either.

6.4 Endogenous Network Formation

So far our analysis focused on characterizing Pareto efficient risk-sharing arrangements subject to local information constraints on an exogenously given network, implicitly assuming that the network structure is mainly shaped by predetermined factors such as kinship. Here we briefly discuss some implications of allowing for endogenous link formation in the context of informal risk sharing with local information constraints. The approach we take is similar as in [Ambrus, Chandrasekhar, and Elliott \(2015\)](#), who consider network formation in a risk-sharing framework with global information contracts, and propose a two-stage game in which in the first stage individuals can simultaneously indicate other individuals they want to link with. If two individuals each indicated each other, the link is formed, and the two connecting individuals each incur a cost of $c \geq 0$.³⁴ The solution concept we use is pairwise stability. In the second stage, whatever network is formed in the first stage, it is assumed that individuals agree on a Pareto efficient risk-sharing arrangement subject to local information constraints.

In our analysis of the CARA-normal framework so far, state independent transfers played a very limited role. However, when we allow for endogenous network formation, it becomes crucial how the network structure influences state independent transfers, and hence the distribution of surplus created by risk sharing, as it directly affects incentives to form links. Therefore, it is important to specify exactly which Pareto efficient risk-sharing arrangement prevails for each possible network that can form. Different ways of specifying state-independent transfers can lead to very different conclusions regarding network formation, as we demonstrate below.

A benchmark case is when all state-independent transfers are set to 0, which case is extensively investigated by [Gao and Moon \(2016\)](#) who assume local equal

³⁴This simple game of network formation was originally considered in [Myerson \(1991\)](#). See also [Jackson and Wolinsky \(1996\)](#).

sharing with no state-independent transfers as an ad hoc sharing rule. They show that, even with zero cost of linking, an individual i 's benefit for establishing an extra link with j falls very fast with the existing number of links the individual i has, as with more existing neighbors (larger d_i) the marginal reduction in self-endowment exposure $\left(\frac{1}{d_i+1} - \frac{1}{d_i+2}\right)$ is small relative to the additional exposure to j 's endowment $\frac{1}{d_j+2}$. Typically this implies severe underinvestment into social links.

An alternative approach is pursued by [Ambrus, Chandrasekhar, and Elliott \(2015\)](#), in the context of risk-sharing arrangements with global information: they assume that the profile of state-independent transfers is determined according to the Myerson value. The Myerson value, proposed in [Myerson \(1980\)](#), is a network-specific version of the Shapley value that allocates surplus according to average incremental contribution of individuals to total social surplus.³⁵ [Ambrus, Chandrasekhar, and Elliott \(2015\)](#) in particular show that with state-independent transfers specified as above (for whatever network is formed), if individuals are ex ante symmetric then there is never underinvestment, that is given any stable network, there is no potential link that is not established, even though its net social value would be strictly positive. Below we show that the same conclusion holds in our setting with local information constraints, in the case of CARA utilities and independently an jointly normally distributed endowments. The detailed specification and the proof are available in Appendix B.6.

Proposition 9. *Suppose that, for any given network structure, the Pareto efficient consumption plan subject to the local information constraint is implemented, and the state-independent transfers are induced by the Myerson values. Consider the first-stage network formation game in which each individual pays a private cost of c for each of her established links. Then, there is no underinvestment in social links in any pairwise stable network.*

We leave a more detailed investigation of network formation in the context of risk sharing with local information constraints to future research.

³⁵[Ambrus, Chandrasekhar, and Elliott \(2015\)](#) also provide micro-foundations, in the form of a decentralized bargaining procedure between neighboring individuals that leads to state independent transfers achieving the Myerson value allocation.

7 Conclusion

This paper analyzes informal risk sharing arrangements assuming that only direct neighbors may observe each others' endowment realizations, and that bilateral transfers must depend on commonly observed information only. Relative to previous models that propose alternative explanations for the inefficiency of these informal insurance arrangements, our framework provides a number of new and testable predictions. First of all, the model provides closed-form expressions for the set of bilateral exchanges that obtain a Pareto efficient allocation, for any network structure. We find that centrally located individuals become quasi insurance providers to more peripheral households. Further, the current setup formalizes, and indeed generalizes, the notion of a “local sharing group” that has been invoked recently in the risk-sharing tests performed in the development literature. We also show that the performance of these risk sharing arrangements is highly sensitive to the type of correlation structures that we assume, and in particular to the assumption that correlations decrease over network distance. This can potentially help explain why informal risk sharing works better in certain settings than in others.

The model provides numerous implications for empirical work. In a first approach, [Milán \(2016\)](#) shows that the current framework fits the observed sharing behavior of indigenous communities in the Bolivian Amazon. However, further empirical work is needed to distinguish local information constraints from other similar contractual frictions, such as the hidden income model identified by [Kinnan \(2011\)](#) as the relevant friction in Thai data. Indeed, in future work we plan to derive a dynamic version of the model that provides testable predictions between current consumption and past information, which can be compared to those of other proposed risk-sharing frictions. Another empirical project to follow from this work takes the model's predictions on bilateral exchanges in order to develop a complete model of spillover effects across individuals that can be used to structurally estimate the underlying network structure following techniques in [Manresa \(2016\)](#).

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Supplement to “Informal Risk Sharing with Local Information”

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(NOT FOR PUBLICATION)

Appendix A: Main Proofs

The proofs for all the lemmas stated below are available in Appendix B.1.

Define $J(t) := \mathbb{E} \left[\sum_{k \in N} \lambda_k u_k \left(e_k - \sum_{h \in N_k} t_{kh} \right) \right]$, the objective function in equation (3).

Lemma 1. \mathcal{T} with $\langle \cdot, \cdot \rangle$ forms an inner product space.

Lemma 2. J is concave on \mathcal{T} .

Lemma 3. J is Gâteaux-differentiable.

Lemma 4. For any $t \in \mathcal{T}$ that solves (4), we have $J'(t) = \mathbf{0}$.

Lemma 5. The set of consumption plan induced by the profiles of transfer rules t in \mathcal{T} is convex.

Proof of Proposition 1

Proof. We first prove the “only if” part. Note that, given any $t \in \mathcal{T}^*$, $\forall i, j$,

$$\begin{aligned} \mathbb{E} \left[\sum_{k \in N} \lambda_k u_k \left(e_k - \sum_{h \in N_k} t_{kh} \right) \right] &= \mathbb{E} \left[\mathbb{E} \left[\sum_{k \in N} \lambda_k u_k \left(e_k - \sum_{h \in N_k} t_{kh} \right) \middle| I_{ij} \right] \right] \\ &\leq \mathbb{E} \left[\max_{t_{ij} \in \mathbb{R}} \mathbb{E} \left[\sum_{k \in N} \lambda_k u_k \left(e_k - \sum_{h \in N_k} t_{kh} \right) \middle| I_{ij} \right] \right] \end{aligned}$$

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This is because, conditional on I_{ij} , t_{ij} must be constant across all possible states, and thus the maximization of the conditional expectation is to solve for the optimal real number t_{ij} . For t to be a solution for problem (3), suppose there exists linked ij such that t_{ij} does not solve the problem (4). Then, by the inequality above, there exists another t_{ij} , specified for each different realization of I_{ij} and hence each possible state of nature, that leads to higher value of $\mathbb{E} \left[\sum_{k \in N} \lambda_k u_k \left(e_k - \sum_{h \in N_k} t_{kh} \right) \right]$, contradicting the optimality of t for problem (3). Note that the “ \mathbb{P} -almost-all” quantifier applies here.

For the “if” part, notice that by Lemma 4, t solves all (4) simultaneously implies that $J'(t) = \mathbf{0}$. As $J : \mathcal{T} \rightarrow \mathbb{R}$ is concave by Lemma 2 and Gâteaux-differentiable by Lemma 3, we can apply a mathematical result on convex optimization in normed space, specifically Theorem 3.24 and Proposition 3.20 in [Peypouquet \(2015\)](#), to conclude that asserting that if $J'(t) = \mathbf{0}$, then $J(t)$ is the unique global maximum. ■

Proof of Proposition 2

Proof. Following the proof of Lemma 2, we can easily show, by the strict concavity of $u_i(\cdot)$, that the objective function in (3) is strictly concave in the consumption plan x . Lemma 5 shows that the set of admissible consumption plan induced by the set of transfer rules in \mathcal{T} is convex. Hence, there is at most of one consumption plan that solves (3). ■

Proof of Corollary 1

Proof. By the concavity (shown in Lemma 2) of the objective function in (4), the FOC is both sufficient and necessary for maximization. The FOC w.r.t t_{ij} , is

$$\mathbb{E} \left[\lambda_i u'_i \left(e_i - \sum_{h \in N_i} t_{ih}(e) \right) + \lambda_j u'_j \left(e_j - \sum_{h \in N_j} t_{jh}(e) \right) \cdot (-1) \middle| I_{ij} \right] = 0$$

Rearranging the above we have

$$\frac{\mathbb{E}_{ij} [u'_i(x_i^t)]}{\mathbb{E}_{ij} [u'_j(x_j^t)]} = \frac{\mathbb{E} [u'_i(e_i - \sum_{h \in N_i} t_{ih}(e)) | I_{ij}]}{\mathbb{E} [u'_j(e_j - \sum_{h \in N_j} t_{jh}(e)) | I_{ij}]} = \frac{\lambda_j}{\lambda_i}.$$

■

Lemma 6. *Given any real vector $c \in \mathbb{R}^n$ such that $\sum_{i \in N} c_i = 0$, there exists a real vector $\mu \in \mathbb{R}^{\sum_i d_i}$ such that $\mu_{ik} + \mu_{ki} = 0$ for every linked pair ik and*

$$\sum_{k \in N_i} \mu_{ik} = c_i.$$

The solution is unique if and only if the network is minimally connected.

Proof of Proposition 3

Proof. Let x_i^* be the consumption plan induced by the transfer t^* described above. Then

$$\begin{aligned} CE(x_i^* | I_{ij}) &= \mathbb{E}_{ij} \left[e_i - \sum_{k \in N_i} t_{ik}^* \right] - \frac{1}{2} r \text{Var}_{ij} \left[e_i - \sum_{k \in N_i} t_{ik}^* \right] \\ &= e_i - \frac{e_i}{d_i + 1} + \frac{e_j}{d_j + 1} - \mu_{ij}^* - \sum_{k \in N_{ij}} \left(\frac{e_i}{d_i + 1} - \frac{e_k}{d_k + 1} + \mu_{ik}^* \right) \\ &\quad - \sum_{k \in N_i \setminus \bar{N}_j} \left(\frac{e_i}{d_i + 1} - \frac{\mathbb{E}_{ij}[e_k]}{d_k + 1} + \mu_{ik}^* \right) - \frac{1}{2} r \text{Var} \left[\sum_{k \in N_i \setminus \bar{N}_j} \frac{e_k}{d_k + 1} \right] \\ &= \frac{e_i}{d_i + 1} + \frac{e_j}{d_j + 1} + \sum_{k \in N_{ij}} \frac{e_k}{d_k + 1} - \sum_{k \in N_i} \mu_{ik}^* - \frac{1}{2} r \sigma^2 \cdot \sum_{k \in N_i \setminus \bar{N}_j} \frac{1}{(d_k + 1)^2}. \end{aligned}$$

The necessary and sufficient condition for t^* to be Pareto efficient is given by (6). Plugging the above into (6) and canceling out the terms dependent on local information $(e_k)_{k \in \bar{N}_{ij}}$, we arrive at the following condition for Pareto efficiency:

$$\sum_{k \in N_i} \mu_{ik}^* + \frac{1}{2} r \sigma^2 \cdot \sum_{k \in N_i \setminus \bar{N}_j} \frac{1}{(d_k + 1)^2} + \frac{1}{r} \ln \lambda_i = \sum_{k \in N_j} \mu_{jk}^* + \frac{1}{2} r \sigma^2 \cdot \sum_{k \in N_j \setminus \bar{N}_i} \frac{1}{(d_k + 1)^2} + \frac{1}{r} \ln \lambda_j \quad (19)$$

Any profile of state-independent transfers μ^* that solves the above system (19) makes t^* efficient under weightings λ .

Notice that, if $CE(x_i^*|I_{ij}) - \frac{1}{r} \ln \lambda_i = CE(x_j^*|I_{ij}) - \frac{1}{r} \ln \lambda_j$ holds for any e ,

$$\begin{aligned} CE(x_i^*) - \frac{1}{r} \ln \lambda_i &= \mathbb{E} \left[CE(x_i^*|I_{ij}) - \frac{1}{r} \ln \lambda_i \right] - \frac{1}{2} r \text{Var} \left[CE(x_i^*|I_{ij}) - \frac{1}{r} \ln \lambda_i \right] \\ &= CE(x_j^*) - \frac{1}{r} \ln \lambda_j \end{aligned}$$

Hence, with G assumed WLOG to be connected, we have

$$\begin{aligned} CE(x_i^*) - \frac{1}{r} \ln \lambda_i &= \frac{1}{n} \sum_{k \in N} \left(CE(x_k^*) - \frac{1}{r} \ln \lambda_k \right) \\ &= -\frac{r\sigma^2}{2n} \sum_{k \in N} \frac{1}{d_k + 1} - \frac{1}{nr} \sum_{k \in N} \ln \lambda_k \end{aligned} \quad (20)$$

On the other hand, as $x_i^* = \frac{e_i}{d_i+1} + \sum_{k \in N_i} \left(\frac{e_k}{d_k+1} - \mu_{ik}^* \right)$,

$$CE(x_i^*) = -\sum_{k \in N_i} \mu_{ik}^* - \frac{1}{2} r \sigma^2 \sum_{k \in \bar{N}_i} \frac{1}{(d_k + 1)^2} \quad (21)$$

Equating the expressions for $CE(x_i^*)$ in (20) and (21), we obtain

$$\sum_{k \in N_i} \mu_{ik}^* = \frac{1}{2} r \sigma^2 \left(\frac{1}{n} \sum_{k \in N} \frac{1}{d_k + 1} - \sum_{k \in \bar{N}_i} \frac{1}{(d_k + 1)^2} \right) + \frac{1}{r} \left(\frac{1}{n} \sum_{k \in N} \ln \lambda_k - \ln \lambda_i \right). \quad (22)$$

Lemma 6 has established that there indeed exists a solution μ^* to (22). Given any solution μ^* to (22), as $\bar{N}_i \setminus (N_i \setminus \bar{N}_j) = \bar{N}_{ij}$, we have

$$\begin{aligned} &\sum_{k \in N_i} \mu_{ik}^* + \frac{1}{2} r \sigma^2 \sum_{k \in N_i \setminus \bar{N}_j} \frac{1}{(d_k + 1)^2} + \frac{1}{r} \ln \lambda_i \\ &= \frac{1}{2} r \sigma^2 \left(\frac{1}{n} \sum_{k \in N} \frac{1}{d_k + 1} - \sum_{k \in \bar{N}_{ij}} \frac{1}{(d_k + 1)^2} \right) + \frac{1}{nr} \sum_{k \in N} \ln \lambda_k \\ &= \sum_{k \in N_j} \mu_{jk}^* + \frac{1}{2} r \sigma^2 \sum_{k \in N_j \setminus \bar{N}_i} \frac{1}{(d_k + 1)^2} + \frac{1}{r} \ln \lambda_j \end{aligned}$$

implying that μ^* also solves the system of equations (19). Hence, t^* is Pareto efficient. ■

Lemma 7. A linear profile of transfer rules $t = (\alpha, \beta, \mu)$ is Pareto efficient if $\forall ij$ s.t. $G_{ij} = 1$,

$$\left\{ \begin{array}{l} \alpha_{ij} = \frac{1}{2} \left(1 - \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} + \sum_{k \in N_{ij}} \beta_{jki} + \gamma_{ij} \right) \\ \beta_{ijk} = \frac{1}{2} \left[\alpha_{ki} - \alpha_{kj} + \sum_{h \in N_{ijk}} (\beta_{ihk} - \beta_{jkh}) \right. \\ \quad \left. - \sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ihk} + \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} + \gamma_{ij} \right] \quad \forall k \in N_{ij} \\ \gamma_{ij} = \frac{\rho}{1+(d_{ij}+1)\rho} \left[\sum_{k \in N_i \setminus \bar{N}_j} (\alpha_{ki} - \sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh}) \right. \\ \quad \left. - \sum_{k \in N_j \setminus \bar{N}_i} (\alpha_{kj} - \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh}) \right. \\ \quad \left. - \sum_{k \in N_{ij}} \left(\sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh} - \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} \right) \right] \end{array} \right. \quad (23)$$

Proof of Proposition 4

Proof. We begin by proving the first part, which only involves triads. We rewrite (10) in the following way:

$$\left\{ \begin{array}{l} 2\alpha_{ij} + \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} - \gamma_{ij} = 1, \quad \forall G_{ij} = 1 \quad \textcircled{1} \\ \alpha_{ki} - \alpha_{kj} + \gamma_{ij} = 0 \quad \forall k \in N_{ij}, \quad \forall G_{ij} = 1; \quad \textcircled{2} \\ \gamma_{ij} = \frac{\rho}{1+(d_{ij}+1)\rho} \left(\sum_{k \in N_i \setminus \bar{N}_j} \alpha_{ki} - \sum_{k \in N_j \setminus \bar{N}_i} \alpha_{kj} \right), \quad \forall G_{ij} = 1; \quad \textcircled{3} \end{array} \right.$$

In matrix form we write

$$\begin{bmatrix} A \\ M \end{bmatrix} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} b \\ \mathbf{0} \end{pmatrix} \quad \begin{array}{c} \textcircled{1} \wedge \textcircled{3} \\ \textcircled{2} \end{array}$$

where α, γ are both $\sum_i d_i$ -dimensional vectors, A is a $(2 \sum_i d_i) \times (2 \sum_i d_i)$ square matrix, $b := \begin{pmatrix} \mathbf{1}_{\sum_i d_i} \\ \mathbf{0}_{\sum_i d_i} \end{pmatrix}$ is a $(2 \sum_i d_i)$ -dimensional vector, M is a $\left(\sum_{G_{ij}=1} d_{ij} \right) \times (2 \sum_i d_i)$ rectangular matrix, and $\mathbf{0}$ is a $\left(\sum_{G_{ij}=1} d_{ij} \right)$ -dimensional vector. The upper block $A \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = b$ corresponds to equations in $\textcircled{1}$ and $\textcircled{3}$, while the lower block

$M \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \mathbf{0}$ corresponds to equations in $\textcircled{2}$.

Note that $\textcircled{3}_{ij}$ in the definition above is not written in its canonical form, i.e., it is not written in such way that the LHS of the equality sign is a linear combination of unknown variables (α, γ) while the right-hand side (RHS) is a constant scalar. In the following, we interpret any written linear equation to be representative of the underlying canonical form obtained by moving all linear combinations of (α, γ) on the RHS of “=” to the LHS (left-hand side) while moving all constants on the LHS to the RHS. For example, we interpret $\textcircled{3}_{ij}$ to represent a canonical form such that the coefficient before the unknown variable γ_{ij} is 1 and the coefficient before α_{ki} for some $k \in N_i \setminus \bar{N}_j$ to be $-\frac{\rho}{1+(d_{ij}+1)\rho}$.³⁹

Given that the system consisting of $\textcircled{1}$ and $\textcircled{3}$ admit a unique solution, its coefficient matrix A and its augmented matrix $\tilde{A} = [A | b]$ must have full rank $2 \sum_i d_i$. To prove that the unique solution of $\textcircled{1}$ and $\textcircled{3}$ also satisfies $\textcircled{2}$, it suffices to show that the augmented matrix for the system of $\textcircled{1}$, $\textcircled{3}$ and $\textcircled{2}$

$$\left[\begin{array}{c|c} A & b \\ \hline M & \mathbf{0} \end{array} \right]$$

still have rank $2 \sum_i d_i$. We show this by demonstrating the existence of $\sum_{G_{ij}=1} d_{ij}$ nonzero and linearly independent vector $\xi \in \mathbb{R}^{2 \sum_i d_i + \sum_{G_{ij}=1} d_{ij}}$ such that

$$\xi' \left[\begin{array}{cc} A & b \\ \hline M & \mathbf{0} \end{array} \right] = (0, 0, \dots, 0)_{2 \sum_i d_i + 1}.$$

We first fix any linked triad ijk .

Multiplying $\textcircled{3}_{ij}$ (the ij -th equation in $\textcircled{3}$) with $(1 + (d_{ij} + 1)\rho)$, we obtain $[1 + (d_{ij} + 1)\rho] \gamma_{ij} = \rho \left(\sum_{h \in N_i \setminus \bar{N}_j} \alpha_{hi} - \sum_{h \in N_j \setminus \bar{N}_i} \alpha_{hj} \right)$, which is equivalent to

$$[1 + (d_{ij} + 1)\rho] \gamma_{ij} = \rho \left[\sum_{h \in N_i} \alpha_{hi} - \sum_{h \in N_j} \alpha_{hj} - \sum_{h \in N_{ij}} (\alpha_{hi} - \alpha_{hj}) - \alpha_{ji} + \alpha_{ij} \right] \quad \textcircled{4}_{ij}.$$

³⁹This convention should resolve any ambiguity about the signs of coefficients before (α, γ) in all the equations written out thereafter.

Adding $\textcircled{2}_{ijh}$ for all $h \in N_{ij} \setminus \{k\}$ to $\textcircled{4}_{ij}$, we get

$$[1 + (d_{ij} + 1)\rho] \gamma_{ij} = \rho \left[\sum_{h \in N_i} \alpha_{hi} - \sum_{h \in N_j} \alpha_{hj} + (d_{ij} - 1) \gamma_{ij} - \alpha_{ki} + \alpha_{kj} - \alpha_{ji} + \alpha_{ij} \right]$$

which is equivalent to

$$(1 + 2\rho) \gamma_{ij} = \rho \left(\sum_{h \in N_i} \alpha_{hi} - \sum_{h \in N_j} \alpha_{hj} + \alpha_{kj} - \alpha_{ki} + \alpha_{ij} - \alpha_{ji} \right) \quad \textcircled{5}_{ij}.$$

Summing up $\textcircled{5}_{ij}$, $\textcircled{5}_{jk}$, $\textcircled{5}_{ki}$, we have

$$\begin{aligned} (1 + 2\rho) (\gamma_{ij} + \gamma_{jk} + \gamma_{ki}) &= \rho [(\alpha_{kj} - \alpha_{ki} + \alpha_{ij} - \alpha_{ji}) + (\alpha_{ik} - \alpha_{ij} + \alpha_{jk} - \alpha_{kj}) \\ &\quad + (\alpha_{ji} - \alpha_{jk} + \alpha_{ki} - \alpha_{ik})] \\ &= 0 \end{aligned}$$

For $n = 3$ and $\rho > -\frac{1}{2}$, or for $n \geq 4$, we have $1 + 2\rho > 0$ and thus

$$\gamma_{ij} + \gamma_{jk} + \gamma_{ki} = 0. \quad \textcircled{6}_{ijk}$$

Alternatively, taking $\textcircled{1}_{ki} - \textcircled{1}_{kj} + \textcircled{2}_{ijk}$, we obtain $\gamma_{ij} - \gamma_{kj} + \gamma_{ki} = 0$. By $\textcircled{3}_{jk} + \textcircled{3}_{kj}$, we have $\gamma_{jk} + \gamma_{kj} = 0$ and thus

$$\gamma_{ij} + \gamma_{jk} + \gamma_{ki} = 0. \quad \textcircled{7}_{ijk}$$

Then $\textcircled{6}_{ijk} - \textcircled{7}_{ijk}$ leads to the tautology “ $0 = 0$ ”. Let $\xi^{ijk} \in \mathbb{R}^{2 \sum_i d_i + \sum_{G_{ij}=1} d_{ij}}$ be a vector that characterizes all the row operations conducted above. Clearly

$$(\xi^{ijk})' \begin{bmatrix} A & b \\ M & \mathbf{0} \end{bmatrix} = \mathbf{0}'.$$

Notice that we can obtain one ξ^{ijk} for each ordered triad (i, j, k) . Clearly each ξ^{ijk} is nonzero: in particular, the entries of ξ that correspond to equations $\textcircled{1}_{ki}$ and $\textcircled{1}_{kj}$ must be nonzero, $\xi_{\textcircled{1}_{ki}} \neq 0$, $\xi_{\textcircled{1}_{kj}} \neq 0$, because $\textcircled{1}_{ki}, \textcircled{1}_{kj}$ are used to obtain $\textcircled{7}_{ijk}$ and nowhere else.

Fixing k , for a row operation in question $\xi^{i_1 i_2 i_3}$, coefficients corresponding to $\textcircled{1}_{kh}$ for $h \in N_k$ may be nonzero only if $i_3 = k$. Hence, $\{\xi^{i_1 i_2 k} : i_1, i_2 \in N_k, G_{i_1 i_2} = 1\}$ must be linearly independent from $\{\xi^{i_1 i_2 i_3} : i_1, i_2 \in N_k, G_{i_1 i_2} = 1, i_3 \neq k\}$. We now consider $\{\xi^{i_1 i_2 k} : i_1, i_2 \in N_k, G_{i_1 i_2} = 1\}$. Notice that $\textcircled{1}_{ki}, \textcircled{1}_{kj}$ show up in the form of “ $\textcircled{1}_{ki} - \textcircled{1}_{kj}$ ” during the process. Hence, summing up along general cycles⁴⁰ is the only possible type of row operations that can cancel out all coefficients before $\textcircled{1}_{ki}$ for all $i \in N_k$. However, this operation does not lead to the tautology $(\mathbf{0}', 0)$, because the coefficients before $\textcircled{2}_{i_1 i_2 k}, \dots, \textcircled{2}_{i_m i_1 k}$ are all kept nonzero. (Notice that these only show up in $\xi^{i_1 i_2 k}$ in the step leading to $\textcircled{7}_{ijk}$ and nowhere else). Hence, no nontrivial linear combination of $\{\xi^{i_1 i_2 k} : i_1, i_2 \in N_k, G_{i_1 i_2} = 1\}$ is zero, so $\{\xi^{i_1 i_2 k} : i_1, i_2 \in N_k, G_{i_1 i_2} = 1\}$ is linearly independent. In summary, we conclude that $\{\xi^{i_1 i_2 i_3} : i_1, i_2 \in N_k, G_{i_1 i_2} = 1\}$ are linearly independent, so we have established the existence of $\sum_{G_{ij}=1} d_{ij}$ nonzero and linearly independent vector $\xi \in \mathbb{R}^{2\sum_i d_i + \sum_{G_{ij}=1} d_{ij}}$.

We now prove the second part, the statement for cycles of any size. Note that we still have $\textcircled{5}_{ij} : (1 + \rho) \gamma_{ij} = \rho \left(\sum_{h \in N_i} \alpha_{hi} - \sum_{h \in N_j} \alpha_{hj} + \alpha_{ij} - \alpha_{ji} \right)$. Given any cycle $i_1 i_2 \dots i_m i_1$, summing up $\textcircled{5}_{i_1 i_2}, \textcircled{5}_{i_2 i_3}, \dots, \textcircled{5}_{i_m i_1}$, we have

$$(1 + \rho) (\gamma_{i_1 i_2} + \gamma_{i_2 i_3} + \dots + \gamma_{i_m i_1}) = \rho (\alpha_{i_1 i_2} + \dots + \alpha_{i_m i_1} - \alpha_{i_2 i_1} - \dots - \alpha_{i_1 i_m}) \quad \textcircled{10}$$

By $\textcircled{1}_{i_1 i_2} - \textcircled{1}_{i_2 i_1}$ and $\gamma_{ij} + \gamma_{ji} = 0$, we have $\alpha_{i_1 i_2} - \alpha_{i_2 i_1} = \sum_{h \in N_{i_2}} \alpha_{i_2 h} - \sum_{h \in N_{i_1}} \alpha_{i_1 h} + 2\gamma_{i_1 i_2}$. Summing over $i_1 i_2, \dots, i_m i_1$,

$$\alpha_{i_1 i_2} + \dots + \alpha_{i_m i_1} - \alpha_{i_2 i_1} - \dots - \alpha_{i_1 i_m} = 2 (\gamma_{i_1 i_2} + \gamma_{i_2 i_3} + \dots + \gamma_{i_m i_1}) \quad \textcircled{11}$$

Then $\textcircled{10} + \rho \times \textcircled{11}$ gives $(1 - \rho) (\gamma_{i_1 i_2} + \gamma_{i_2 i_3} + \dots + \gamma_{i_m i_1}) = 0$. For $\rho < 1$, we have $\gamma_{i_1 i_2} + \gamma_{i_2 i_3} + \dots + \gamma_{i_m i_1} = 0$. ■

⁴⁰ By general cycles we mean cycles that may involve “self cycles” of the form “ $i_1 i_2 i_1$ ”.

Proof of Proposition 5

Proof. Write system (10) in the following form:

$$\begin{cases} 2\alpha_{ij} + \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} - \gamma_{ij} = 1, & \textcircled{1}_{ij} \quad \forall G_{ij} = 1 \\ \gamma_{ij} = \frac{\rho}{1+(d_{ij}+1)\rho} \left(\sum_{k \in N_i \setminus \bar{N}_j} \alpha_{ki} - \sum_{k \in N_j \setminus \bar{N}_i} \alpha_{kj} \right), & \textcircled{3}_{ij} \quad \forall G_{ij} = 1. \end{cases}$$

This is a system of $2 \sum_i d_i$ equations in $2 \sum_i d_i$ variables (α, γ) . Notice that this system can have at most one solution by Proposition 2, as each distinct solution to the above system will define a distinct consumption plan.

Write system (12) in the following form: $\forall ij$ s.t. $G_{ij} = 1$, and $\forall i \in N$

$$\begin{cases} \alpha_{ji} = \Lambda_j - \frac{\rho}{1-\rho} \left(\sum_{k \in N_i} \alpha_{ki} + \alpha_{ii} \right), & \textcircled{12}_{ji} \quad \forall G_{ij} = 1 \\ \alpha_{ii} = \Lambda_i - \frac{\rho}{1-\rho} \left(\sum_{k \in N_i} \alpha_{ki} + \alpha_{ii} \right), & \textcircled{12}_{ii} \quad \forall i \in N \\ \alpha_{ii} + \sum_{k \in N_i} \alpha_{ik} = 1, & \textcircled{13}_i \quad \forall i \in N \end{cases}$$

This is a system of $(\sum_i d_i + 2n)$ equations in $(\sum_i d_i + 2n)$ variables (α, Λ) . Suppose that this system has a unique solution.⁴¹

We now show that there exist $\sum_i d_i$ linearly independent sequences of row operations that produce the tautology “ $0 = 0$ ”. Given that the system $\textcircled{12}\textcircled{13}$ has a unique solution $((\alpha_{ij})_{G_{ij}=1}, \alpha)$, this will imply that the $(\alpha_{ij})_{G_{ij}=1}$, along with γ defined by $\textcircled{3}$, will also solve system $\textcircled{1}\textcircled{3}$.

Notice that, by the proof of Proposition 4, $\textcircled{1}$ and $\textcircled{3}$ imply that

$$(1 + \rho) \gamma_{ij} = \rho \left(\sum_{h \in N_i} \alpha_{hi} - \sum_{h \in N_j} \alpha_{hj} + \alpha_{ij} - \alpha_{ji} \right) \quad \textcircled{5}_{ij}.$$

In other words, $\textcircled{5}_{ij}$ can be obtained by a sequence of row operations on $\textcircled{1}$ and $\textcircled{3}$.

Consider a fixed linked pair ij with $i < j$.

By $(1 - \rho) \times \left(\textcircled{12}_{ji} - \textcircled{12}_{ij} + \textcircled{12}_{ii} - \textcircled{12}_{jj} \right)$, we have

$$(1 - \rho) (\alpha_{ji} - \alpha_{ij} + \alpha_{ii} - \alpha_{jj}) + 2\rho \left(\sum_{k \in N_i} \alpha_{ki} + \alpha_{ii} - \sum_{k \in N_j} \alpha_{kj} - \alpha_{jj} \right) = 0,$$

⁴¹It indeed has a unique solution given by Proposition 6.

which is equivalent to

$$(1 + \rho)(\alpha_{ii} - \alpha_{jj} + \alpha_{ij} - \alpha_{ji}) + 2\rho \left(\sum_{k \in N_i} \alpha_{ki} - \sum_{k \in N_j} \alpha_{kj} + \alpha_{ij} - \alpha_{ji} \right) = 0.$$

Plugging $\textcircled{5}_{ij}$ into the second term above, we have

$$(1 + \rho)(\alpha_{ii} - \alpha_{jj}) + (1 - \rho)(\alpha_{ji} - \alpha_{ij}) + 2(1 + \rho)\gamma_{ij} = 0,$$

which, divided by $(1 + \rho)$ on both sides, is equivalent to

$$\alpha_{ii} - \alpha_{jj} + \alpha_{ji} - \alpha_{ij} + 2\gamma_{ij} = 0. \textcircled{14}_{ij}$$

By $\textcircled{13}_i - \textcircled{1}_{ij}$, we have

$$\alpha_{ii} - \alpha_{ij} + \gamma_{ij} = 0. \textcircled{15}_{ij}$$

By $\textcircled{13}_j - \textcircled{1}_{ji}$, we have $\alpha_{jj} - \alpha_{ji} + \gamma_{ji} = 0$. As $\textcircled{3}_{ij} + \textcircled{3}_{ji}$ implies $\gamma_{ij} + \gamma_{ji} = 0$, we have

$$\alpha_{jj} - \alpha_{ji} - \gamma_{ij} = 0. \textcircled{16}_{ij}$$

By $\textcircled{14}_{ij} - \textcircled{15}_{ij} + \textcircled{16}_{ij}$, we reach the tautology “ $0 = 0$ ”.

Now, consider $\textcircled{12}_{ji} + \textcircled{12}_{ij} - \textcircled{12}_{ii} - \textcircled{12}_{jj}$, which leads to

$$\alpha_{ji} + \alpha_{ij} - \alpha_{ii} - \alpha_{jj} = 0. \textcircled{17}_{ij}$$

Then $\textcircled{17}_{ij} + \textcircled{15}_{ij} + \textcircled{16}_{ij}$ leads to the tautology “ $0 = 0$ ”.

In summary of the above, for each fixed linked pair ij with $i < j$, we have established that

$$\begin{cases} \xi_{ij} : \frac{1-\rho}{1+\rho} \left(\textcircled{12}_{ji} - \textcircled{12}_{ij} + \textcircled{12}_{ii} - \textcircled{12}_{jj} \right) - \textcircled{13}_i + \textcircled{13}_j + \zeta'_{ij} \textcircled{1} + \eta'_{ij} \textcircled{3} = \mathbf{0}' \\ \tilde{\xi}_{ij} : \textcircled{12}_{ji} + \textcircled{12}_{ij} - \textcircled{12}_{ii} - \textcircled{12}_{jj} + \textcircled{13}_i + \textcircled{13}_j + \tilde{\zeta}'_{ij} \textcircled{1} + \tilde{\eta}'_{ij} \textcircled{3} = \mathbf{0}' \end{cases}$$

for some conformable vector $\zeta_{ij}, \tilde{\zeta}_{ij}, \eta_{ij}, \tilde{\eta}_{ij}$. Clearly, the two tautology-generating row operations above are linear independent: any linear combination of the two operations

that cancels out $\textcircled{12}_{ji}$ cannot cancel out $\textcircled{12}_{ij}$.

Moreover, $\textcircled{12}_{ij}, \textcircled{12}_{ji}$ do not show up in any tautology-generating row operation within $\{\xi_{hk}, \tilde{\xi}_{hk} : (i, j) \neq \{h, k\}\}$, so $\{\xi_{ij}, \tilde{\xi}_{ij}\}$ must be linearly independent from $\{\xi_{hk}, \tilde{\xi}_{hk} : (i, j) \neq \{h, k\}\}$.

Hence, we have constructed a set of $\sum_i d_i$ linearly independent tautology-generating row operations $\{\xi_{ij}, \tilde{\xi}_{ij} : G_{ij} = 1, i < j\}$. ■

Proof of Proposition 6

Proof. Let $\bar{G} := G + \mathbf{I}_n$ so that $\bar{G}_{ii} = 1 \forall i \in N$. The optimality conditions given in equation (12.1) and (12.2) can be rewritten as

$$\alpha_{ji} = \bar{G}_{ij} \left(\Lambda_j - \frac{\rho}{1-\rho} \sum_{k \in N} \bar{G}_{ik} \alpha_{ki} \right) \quad (24)$$

Let $\bar{\alpha}_i := (\alpha_{1i}, \alpha_{2i}, \dots, \alpha_{ni})'$ denote the vector of i 's inflow shares, $\mathbf{\Lambda} = (\Lambda_1, \Lambda_2, \dots, \Lambda_n)'$ the vector of rescaled constraint multipliers, and g_i represent the i -th column of \bar{G} . Then (24) can be rewritten in vector form as

$$\left(\mathbf{I} + \frac{\rho}{1-\rho} g_i g_i' \right) \alpha_i = \text{diag}(g_i) \mathbf{\Lambda}$$

where $\text{diag}(g_i)$ is a diagonal matrix with g_i 's entries on the diagonal. Left-multiplying both sides by $\left(\mathbf{I} - \frac{\rho}{1+\rho d_i} g_i g_i' \right)$, which is well-defined for any $\rho > -\frac{1}{n-1}$ and any G , we have

$$\alpha_i = \left(\mathbf{I} - \frac{\rho}{1+\rho d_i} g_i g_i' \right) \text{diag}(g_i) \mathbf{\Lambda}$$

As $g_i g_i' \cdot \text{diag}(g_i) = g_i g_i'$, the above becomes

$$\alpha_i = \left(\text{diag}(g_i) - \frac{\rho}{1+\rho d_i} g_i g_i' \right) \mathbf{\Lambda} \quad (25)$$

Now, notice that (12.3) implies

$$1 = \sum_{j \in N} \alpha_{ij} = (d_i + 1) \Lambda_i - \sum_{j \in N} \bar{G}_{ij} \left(\frac{\rho}{1+\rho d_j} \sum_k \bar{G}_{jk} \Lambda_k \right) \quad (26)$$

and thus we have

$$\Lambda_i = \frac{1}{d_i + 1} \left(1 + \sum_{j \in \bar{N}_i} \sum_{k \in \bar{N}_j} \frac{\rho}{1 + \rho d_j} \Lambda_k \right).$$

This establishes the recursive representation of the solution.

To obtain the closed-form solution, rewrite equation (26) as

$$\mathbf{1} = \sum_{i \in N} \left(\text{diag}(g_i) - \frac{\rho}{1 + \rho d_i} g_i g_i' \right) \Lambda = (\bar{D} - \bar{G}\Psi\bar{G}) \Lambda$$

where \bar{D} is a diagonal matrix with its i -th diagonal entry being $d_i + 1$, and Ψ is a diagonal matrix with its i -th diagonal entry being $\frac{\rho}{1 + \rho d_i}$. Notice that $\forall \xi \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,

$$\begin{aligned} \xi' (\bar{D} - \bar{G}\Psi\bar{G}) \xi &= \sum_{i \in N} (d_i + 1) \xi_i^2 - \sum_{i \in N} \frac{\rho}{1 + \rho d_i} \left(\sum_{j \in \bar{N}_i} \xi_j \right)^2 \\ &\geq \sum_{i \in N} (d_i + 1) \xi_i^2 - \sum_{i \in N} \frac{1}{1 + d_i} \left(\sum_{j \in \bar{N}_i} \xi_j \right)^2 \\ &\geq \sum_{i \in N} (d_i + 1) \xi_i^2 - \sum_{i \in N} \frac{1}{1 + d_i} \cdot (1 + d_i) \sum_{j \in \bar{N}_i} \xi_j^2 \\ &= \sum_{i \in N} (d_i + 1) \xi_i^2 - \sum_{i \in N} (d_i + 1) \xi_i^2 \\ &= 0 \end{aligned}$$

where the equality holds if and only if $\rho = 1$ and $\xi = c \cdot \mathbf{1}$ for some $c > 0$. Hence, $\forall \rho \in (-\frac{1}{n-1}, 1)$, $(\bar{D} - \bar{G}\Psi\bar{G})$ is positive definite and thus invertible. Hence,

$$\begin{aligned} \Lambda &= (\bar{D} - \bar{G}\Psi\bar{G})^{-1} \mathbf{1}, \\ \alpha_i &= \left(\text{diag}(g_i) - \frac{\rho}{1 + \rho d_i} g_i g_i' \right) (\bar{D} - \bar{G}\Psi\bar{G})^{-1} \mathbf{1}. \end{aligned}$$

Finally, we solve for the inverse matrix above as a series of powers of \bar{G} . Notice that

$$(\bar{D} - \bar{G}\Psi\bar{G})^{-1} = \left(\bar{D}^{\frac{1}{2}} \left(\mathbf{I} - \bar{D}^{-\frac{1}{2}} \bar{G}\Psi\bar{G}\bar{D}^{-\frac{1}{2}} \right) \bar{D}^{\frac{1}{2}} \right)^{-1} = \bar{D}^{-\frac{1}{2}} \left(\mathbf{I} - \bar{D}^{-\frac{1}{2}} \bar{G}\Psi\bar{G}\bar{D}^{-\frac{1}{2}} \right)^{-1} \bar{D}^{-\frac{1}{2}}$$

where the middle term $\left(\mathbf{I} - \bar{D}^{-\frac{1}{2}}\bar{G}\Psi\bar{G}\bar{D}^{-\frac{1}{2}}\right)$ is also invertible and positive definite for $\rho \in \left(-\frac{1}{n-1}, 1\right)$ due to the positive definiteness of $\bar{D} - \bar{G}\Psi\bar{G}$ and the invertibility of \bar{D} . For $\rho \in (0, 1)$, notice that $\bar{D}^{-\frac{1}{2}}\bar{G}\Psi\bar{G}\bar{D}^{-\frac{1}{2}}$ is also positive definite, so its eigenvalues must be positive. Also, its largest eigenvalue φ_{max} must be smaller than 1. Otherwise, there exists a nonzero vector ξ such that

$$\xi' \left(\mathbf{I} - \bar{D}^{-\frac{1}{2}}\bar{G}\Psi\bar{G}\bar{D}^{-\frac{1}{2}} \right) \xi = (1 - \varphi_{max}) \xi' \xi < 0$$

contradicting the positive definiteness of $\left(\mathbf{I} - \bar{D}^{-\frac{1}{2}}\bar{G}\Psi\bar{G}\bar{D}^{-\frac{1}{2}}\right)$. Then, we may write

$$\left(\mathbf{I} - \bar{D}^{-\frac{1}{2}}\bar{G}\Psi\bar{G}\bar{D}^{-\frac{1}{2}}\right)^{-1} = \mathbf{I} + \sum_{k=1}^{\infty} \left(\bar{D}^{-\frac{1}{2}}\bar{G}\Psi\bar{G}\bar{D}^{-\frac{1}{2}}\right)^k$$

and thus

$$\begin{aligned} (\bar{D} - \bar{G}\Psi\bar{G})^{-1} &= \bar{D}^{-1} + \bar{D}^{-\frac{1}{2}} \sum_{k=1}^{\infty} \left(\bar{D}^{-\frac{1}{2}}\bar{G}\Psi\bar{G}\bar{D}^{-\frac{1}{2}}\right)^k \bar{D}^{-\frac{1}{2}} \\ &= \bar{D}^{-1} + \sum_{k=1}^{\infty} \left(\bar{D}^{-1}\bar{Q}\right)^k \bar{D}^{-1} \end{aligned}$$

where $\bar{Q} := \bar{G}\Psi\bar{G}$ can be interpreted as the weighted square of the extended adjacency matrix. Consider the set of all paths of length q between i and j under G as

$$\Pi_{ij}^q(G) = \{(i_0, i_1, i_2, \dots, i_q) \mid i_0 = i, i_q = j \text{ and } \bar{G}_{i_n i_{n+1}} = 1 \text{ for } n = 0, 1, \dots, q-1\}$$

For every $\pi_{ij} \in \Pi_{ij}^q(G)$, let $W(\pi_{ij})$ denote the weight associated to this path. It is not difficult to see that,

$$W(\pi_{ij}) = \frac{1}{d_i + 1} \frac{\rho}{1 + \rho d_{i_1}} \frac{1}{d_{i_2} + 1} \frac{\rho}{1 + \rho d_{i_3}} \cdots \frac{1}{d_j + 1}$$

Then

$$\begin{aligned}
\Lambda_i &= \left[(\bar{D} - \bar{G}\Psi\bar{G})^{-1} \mathbf{1} \right]_i \\
&= \left(\bar{D}^{-1} \mathbf{1} \right)_i + \left(\sum_{k=1}^{\infty} (\bar{D}^{-1} \bar{Q})^k \bar{D}^{-1} \mathbf{1} \right)_i \\
&= \frac{1}{d_i + 1} + \left[\sum_{k=1}^{\infty} (\bar{D}^{-1} \bar{G}\Psi\bar{G})^k \right]_i \bar{D}^{-1} \mathbf{1} \\
&= \frac{1}{d_i + 1} + \sum_{j \in N} \left[\sum_{k=1}^{\infty} (\bar{D}^{-1} \bar{G}\Psi\bar{G})^k \right]_{ij} \cdot \frac{1}{d_j + 1} \\
&= \frac{1}{d_i + 1} + \sum_{j \in N} \sum_{k=1}^{\infty} (\bar{D}_{[i]}^{-1} \bar{G}\Psi\bar{G} \bar{D}^{-1} \dots \bar{D}^{-1} \bar{G}\Psi\bar{G}_{(j)}) \frac{1}{d_j + 1} \\
&= \frac{1}{d_i + 1} + \sum_{j \in N} \sum_{q=1}^{\infty} \sum_{\pi_{ij} \in \Pi_{ij}^{2q}} \left(\frac{1}{d_i + 1} \cdot \frac{\rho}{1 + \rho d_{i_1}} \cdot \frac{1}{d_{i_2} + 1} \cdot \dots \right) \frac{1}{d_j + 1} \\
&= \frac{1}{d_i + 1} + \sum_{q=1}^{\infty} \sum_{j \in N} \sum_{\pi_{ij} \in \Pi_{ij}^{2q}} W(\pi_{ij})
\end{aligned}$$

This concludes the proof. ■

Proof of Proposition 7

Proof. (i) is immediate from (10.2). (ii) follows from differencing (13). (iii) follows from $\textcircled{7}_{ij}$ in the proof of Proposition 5. ■

Appendix B: Supplementary Materials

B.1 Proofs of Lemmas

Lemma 1. \mathcal{T}^* with $\langle \cdot, \cdot \rangle$ forms an inner product space.

Proof. We first show that $\langle \cdot, \cdot \rangle$ is a well-defined inner product. Symmetry immediately follows from the definition. Linearity in the first argument follows from the linearity of the expectation operator:

$$\begin{aligned} \langle \alpha s + \beta t, r \rangle &= \mathbb{E} \left[\sum_{G_{ij}=1} (\alpha s_{ij} + \beta t_{ij}) r_{ij} \right] = \alpha \mathbb{E} \left[\sum_{G_{ij}=1} s_{ij} r_{ij} \right] + \beta \mathbb{E} \left[\sum_{G_{ij}=1} t_{ij} r_{ij} \right] \\ &= \alpha \langle s, r \rangle + \beta \langle t, r \rangle. \end{aligned}$$

Positive definiteness is also obvious: $\langle t, t \rangle = \mathbb{E} \left[\sum_{G_{ij}=1} t_{ij}^2(e) \right] \geq 0$ and $\langle t, t \rangle = 0$ if and only if $t = \mathbf{0}$, i.e., $t_{ij}(\omega) = 0$ for all linked ij and \mathbb{P} -almost all $e \in \Omega$.

We then show that \mathcal{T} is a linear space. $\forall s, t \in \mathcal{T}, \forall \alpha, \beta \in \mathbb{R}, \alpha s(I_{ij}) + \beta t(I_{ij})$ is also $\sigma(I_{ij})$ -measurable, and

$$\alpha s_{ij}(e) + \beta t_{ij}(e) = -(\alpha s_{ji}(e) + \beta t_{ji}(e)).$$

Finiteness of expectation is obvious. Hence, $\alpha s + \beta t \in \mathcal{T}$. ■

Lemma 2. The objective function in (3)

$$J(t) := \mathbb{E} \left[\sum_{k \in N} \lambda_k u_k \left(e_k - \sum_{h \in N_k} t_{kh}(e) \right) \right]$$

is concave on \mathcal{T} .

Proof. $\forall s, t \in \mathcal{T}, \forall \alpha \in [0, 1],$

$$\begin{aligned}
& J(\alpha s + (1 - \alpha)t) \\
&= \mathbb{E} \left[\sum_i \lambda_i u_i \left(e_i - \sum_{j \in N_i} (\alpha s_{ij}(e) + (1 - \alpha)t_{ij}(e)) \right) \right] \\
&= \sum_i \lambda_i \mathbb{E} \left[u_i \left(\alpha \left(e_i - \sum_{j \in N_i} s_{ij}(e) \right) + (1 - \alpha) \left(e_i - \sum_{j \in N_i} t_{ij}(e) \right) \right) \right] \\
&\geq \sum_i \lambda_i \mathbb{E} \left[\alpha u_i \left(e_i - \sum_{j \in N_i} s_{ij}(e) \right) + (1 - \alpha) u_i \left(e_i - \sum_{j \in N_i} t_{ij}(e) \right) \right] \\
&= \alpha \mathbb{E} \left[\sum_i \lambda_i u_i \left(e_i - \sum_{j \in N_i} s_{ij}(e) \right) \right] + (1 - \alpha) \mathbb{E} \left[\sum_i \lambda_i u_i \left(e_i - \sum_{j \in N_i} t_{ij}(e) \right) \right] \\
&= \alpha J(s) + (1 - \alpha) J(t).
\end{aligned}$$

■

Lemma 3. *J is Gâteaux-differentiable.*

Proof. $\forall s, t \in \mathcal{T}$, for $\alpha > 0$,

$$\begin{aligned}
& \frac{J(t + \alpha s) - J(t)}{\alpha} \\
&= \mathbb{E} \left[\sum_i \lambda_i \left[\frac{u_i \left(e_i - \sum_{j \in N_i} t_{ij}(e) - \alpha \sum_{j \in N_i} s_{ij}(e) \right) - u_i \left(e_i - \sum_{j \in N_i} t_{ij}(e) \right)}{\alpha} \right] \right] \\
&= \mathbb{E} \left[\sum_i \lambda_i \left[\frac{u'_i \left(e_i - \sum_{j \in N_i} t_{ij}(e) - \alpha \tilde{s}(e) \right) \cdot \alpha \sum_{j \in N_i} s_{ij}(e)}{\alpha} \right] \right] \\
&\hspace{15em} \text{for some } \tilde{s}_{ij}(e) \text{ between } 0 \text{ and } \sum_{j \in N_i} s_{ij}(e) \\
&= -\mathbb{E} \left[\sum_i \lambda_i \left[u'_i \left(e_i - \sum_{j \in N_i} t_{ij}(e) - \alpha \tilde{s} \right) \cdot \sum_{j \in N_i} s_{ij}(e) \right] \right] \\
&\rightarrow -\mathbb{E} \left[\sum_i \lambda_i \left[u'_i \left(e_i - \sum_{j \in N_i} t_{ij}(e) \right) \cdot \sum_{j \in N_i} s_{ij}(e) \right] \right] \text{ as } \alpha \rightarrow 0 \\
&= -\sum_i \lambda_i \mathbb{E} \left[u'_i \left(e_i - \sum_{j \in N_i} t_{ij}(e) \right) \mathbf{1}_{i \times N_i} \cdot s(e) \right] \\
&= \sum_i \lambda_i \langle f_i, s \rangle
\end{aligned}$$

where

$$f_i(e) := -u'_i \left(e_i - \sum_{j \in N_i} t_{ij}(e) \right) \mathbf{1}_{i \times N_i}$$

and $\mathbf{1}_{i \times N_i}$ is vector of 0 and 1s that equals 1 for the (directed) link ij for any $j \in N_i$ so that $\mathbf{1}_{i \times N_i} \cdot s(e) = \sum_{j \in N_i} s_{ij}(e)$. Define $J'(t) : \mathcal{T} \rightarrow \mathbb{R}$ by

$$J'(t) s = \sum_i \lambda_i \langle f_i, s \rangle.$$

Clearly $J'(t)$ is a linear operator on \mathcal{T} , and is thus the Gâteaux-derivative of J . ■

Lemma 4. *For any $t \in \mathcal{T}$ that solves (4), we have*

$$J'(t) = \mathbf{0}.$$

Proof. To solve (4)

$$\max_{\tilde{t}_{ij} \in \mathbb{R}} J^{(ij, I_{ij})}(\tilde{t}_{ij}) := \mathbb{E} \left[\lambda_i u_i \left(e_i - \tilde{t}_{ij} - \sum_{h \in N_i} t_{ih} \right) + \lambda_j u_j \left(e_j + \tilde{t}_{ij} - \sum_{h \in N_j} t_{jh} \right) \middle| I_{ij} \right]$$

we first notice the objective function $J^{(ij, I_{ij})}(\tilde{t}_{ij})$ is strictly concave in \tilde{t}_{ij} on \mathbb{R} . Hence, the sufficient and necessary condition for optimality is given by the FOC:

$$\mathbb{E} \left[\lambda_i u'_i \left(e_i - \sum_{h \in N_i} t_{ih}(e) \right) \middle| I_{ij} \right] = \mathbb{E} \left[\lambda_j u'_j \left(e_j - \sum_{h \in N_j} t_{jh}(e) \right) \middle| I_{ij} \right]$$

Then, $\forall s \in \mathcal{T}$,

$$\begin{aligned} J'(t) s &= -\mathbb{E} \left[\sum_i \lambda_i \left[u'_i \left(e_i - \sum_{j \in N_i} t_{ij}(e) \right) \cdot \sum_{j \in N_i} s_{ij}(e) \right] \right] \\ &= -\frac{1}{2} \sum_{G_{ij}=1} \mathbb{E} \left[\left(\lambda_i u'_i \left(e_i - \sum_{h \in N_i} t_{ih}(e) \right) - \lambda_j u'_j \left(e_j - \sum_{h \in N_j} t_{jh}(e) \right) \right) \cdot s_{ij}(e) \right] \\ &= -\frac{1}{2} \sum_i \sum_{j \in N_i} \mathbb{E} \left[s_{ij}(I_{ij}) \cdot \mathbb{E} \left[\lambda_i u'_i \left(e_i - \sum_{h \in N_i} t_{ih}(e) \right) - \lambda_j u'_j \left(e_j - \sum_{h \in N_j} t_{jh}(e) \right) \middle| I_{ij} \right] \right] \\ &= -\frac{1}{2} \sum_i \sum_{j \in N_i} \mathbb{E} [s_{ij}(I_{ij}) \cdot 0] \\ &= 0. \end{aligned}$$

Hence $J'(t) = \mathbf{0}$. ■

Lemma 5. *The set of consumption plan induced by the profiles of transfer rules t in \mathcal{T} is convex.*

Proof. Let x, x' be two profiles of consumption plans induced by t, t' respectively. Then $\forall \lambda \in [0, 1]$,

$$\begin{aligned} \lambda x_i(e) + (1 - \lambda) x'_i(e) &= \lambda \left[e_i - \sum_{j \in N_i} t_{ij}(e) \right] + (1 - \lambda) \left[e_i - \sum_{j \in N_i} t'_{ij}(e) \right] \\ &= e_i - \sum_{j \in N_i} \left[\lambda t_{ij}(e) + (1 - \lambda) t'_{ij}(e) \right] \end{aligned}$$

Thus $(\lambda x + (1 - \lambda) x')$ can be induced by $(\lambda t + (1 - \lambda) t')$. \mathcal{T} , as an inner product space, is convex, so the set of consumption plans induced by the profiles of transfer rules in \mathcal{T} must also be convex. ■

Lemma 6. *Given any real vector $c \in \mathbb{R}^n$ such that $\sum_{i \in N} c_i = 0$, there exists a real vector $\mu \in \mathbb{R}^{\sum_i d_i}$ such that $\mu_{ik} + \mu_{ki} = 0$ for every linked pair ik and*

$$\sum_{k \in N_i} \mu_{ik} = c_i. \quad (27)$$

The solution is unique if and only if the network is minimally connected.

Proof. With the restrictions that $\mu_{ik} = -\mu_{ki}$ for all linked pair ik , (27) constitutes a system of n linear equations with $\frac{1}{2} \sum_{i \in N} d_i$ variables μ_{ik} . Summing up all the n equations, we have

$$0 = \sum_{i < k, G_{ik}=1} (\mu_{ik} + \mu_{ki}) = \sum_{i \in N} c_i = 0.$$

Hence, the n linear equations impose at most $(n - 1)$ linearly independent conditions.

Viewing (27) in vector form,

$$C\mu = c$$

where C is a $n \times \frac{1}{2} \sum_{i \in N} d_i$ matrix. Note that in each column of C , denoted C_{ij} for $i < j$, there are either no nonzero entries (when $G_{ij} = 0$), or just two nonzero entries: 1 on the i -th row and -1 on the j -th row when $G_{ij} = 1$. Suppose $G_{ij} = 1$. Then, given any subset of individuals S that include i and j , if the rows of C corresponding to S are linearly dependent, these rows must sum to 0: this can be true only if all entries ik with $i \in S$ and $k \notin S$ are zero, implying that S form a component under G , and thus G is not connected if $\#(S) < n$. This is in contradiction with the supposition that G is connected when $\#(S) < n$. Hence, C must have exactly $(n - 1)$ linearly independent rows.

Let \tilde{C} and \tilde{c} be the first $(n - 1)$ rows of C and c . Then, as \tilde{C} has full row rank, there always exists a solution to $\tilde{C}\mu = \tilde{c}$, and any of the solutions μ must also solve the equation $C\mu = c$. The solution is unique if and only if the component is minimally connected, when there are precisely $(n - 1)$ links and thus \tilde{C} is an invertible square matrix.

We can obtain one particular solution using the following algorithm. First, we can arbitrarily select a subset of links that minimally connect the nodes, i.e., the

graph restricted to this subset of links is minimally connected. Then, there must exist at least one peripheral node, and we can first easily obtain μ_{ij} for all such peripheral nodes $i \in P_1 := \{k \in N : d_k = 1\}$. Then, we can look for new peripheral nodes ignoring the links involving nodes in P_1 , and obtain μ_{ij} for all $i \in P_2 := \{k \in N : k \notin P_1 \wedge G_{kj} = 1 \text{ for some } j \in P_1\}$ with all previously calculated μ 's taken as given. We iterate this process until we exhaust all nodes. Then we are left with a profile of μ that solves (27). ■

Lemma 7. *A linear profile of transfer rules $t = (\alpha, \beta, \mu)$ is Pareto efficient if $\forall ij$ s.t. $G_{ij} = 1$,*

$$\left\{ \begin{array}{l} \alpha_{ij} = \frac{1}{2} \left(1 - \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} + \sum_{k \in N_{ij}} \beta_{jki} + \gamma_{ij} \right) \\ \beta_{ijk} = \frac{1}{2} \left[\alpha_{ki} - \alpha_{kj} + \sum_{h \in N_{ijk}} (\beta_{ihk} - \beta_{jhk}) \right. \\ \quad \left. - \sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ihk} + \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} + \gamma_{ij} \right] \quad \forall k \in N_{ij} \\ \gamma_{ij} = \frac{\rho}{1+(d_{ij}+1)\rho} \left[\sum_{k \in N_i \setminus \bar{N}_j} (\alpha_{ki} - \sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh}) \right. \\ \quad \left. - \sum_{k \in N_j \setminus \bar{N}_i} (\alpha_{kj} - \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh}) \right. \\ \quad \left. - \sum_{k \in N_{ij}} (\sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh} - \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh}) \right] \end{array} \right. \quad (23)$$

Proof. For each $k \in N_i \setminus \{j\}$, we then have

$$\begin{aligned} \sum_{k \in N_i \setminus \{j\}} t_{ik} &= e_i \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} - \sum_{k \in N_i \setminus \{j\}} \alpha_{ki} e_k + \sum_{k \in N_i \setminus \{j\}} \sum_{h \in N_{ik}} \beta_{ikh} e_h + c_{ij} \\ &= e_i \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} - \sum_{k \in N_{ij}} \alpha_{ki} e_k + \sum_{k \in N_{ij}} \left(\beta_{ikj} e_j + \sum_{h \in N_{ijk}} \beta_{ikh} e_h \right) + \sum_{k \in N_i \setminus \bar{N}_j} \sum_{h \in N_{ijk}} \beta_{ikh} e_h \\ &\quad - \sum_{k \in N_i \setminus \bar{N}_j} \alpha_{ki} e_k + \sum_{k \in N_{ij}} \sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh} e_h + \sum_{k \in N_i \setminus \bar{N}_j} \sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh} e_h + c_{ij} \end{aligned}$$

so that

$$\begin{aligned}
t_{ij} &= \frac{1}{2}e_i - \frac{1}{2}e_j - \frac{1}{2} \sum_{k \in N_i \setminus \{j\}} \alpha_{ik}e_i + \frac{1}{2} \sum_{k \in N_j \setminus \{i\}} \alpha_{jk}e_j - \frac{1}{2} \sum_{k \in N_{ij}} (\beta_{ikj}e_j - \beta_{jki}e_i) \\
&\quad + \frac{1}{2} \sum_{k \in N_{ij}} \left[(\alpha_{ki} - \alpha_{kj})e_k - \sum_{h \in N_{ijk}} (\beta_{ikh} - \beta_{jkh})e_h \right] \\
&\quad - \frac{1}{2} \sum_{k \in N_i \setminus \bar{N}_j} \sum_{h \in N_{ijk}} \beta_{ikh}e_h + \frac{1}{2} \sum_{k \in N_j \setminus \bar{N}_i} \sum_{h \in N_{ijk}} \beta_{jkh}e_h \\
&\quad - \frac{1}{2} \frac{\rho}{1 + (d_{ij} + 1)\rho} \left(e_i + e_j + \sum_{k \in N_{ij}} e_k \right) \sum_{k \in N_{ij}} \left(\sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh} - \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} \right) \\
&\quad + \frac{1}{2} \frac{\rho}{1 + (d_{ij} + 1)\rho} \left(e_i + e_j + \sum_{k \in N_{ij}} e_k \right) \sum_{k \in N_i \setminus \bar{N}_j} \left(\alpha_{ki} - \sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh} \right) \\
&\quad - \frac{1}{2} \frac{\rho}{1 + (d_{ij} + 1)\rho} \left(e_i + e_j + \sum_{k \in N_{ij}} e_k \right) \sum_{k \in N_j \setminus \bar{N}_i} \left(\alpha_{kj} - \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} \right) \\
&= \frac{1}{2} \left\{ 1 - \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} + \sum_{k \in N_{ij}} \beta_{jki} + \frac{\rho}{1 + (d_{ij} + 1)\rho} \left[\sum_{k \in N_i \setminus \bar{N}_j} \left(\alpha_{ki} - \sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh} \right) \right. \right. \\
&\quad \left. \left. - \sum_{k \in N_j \setminus \bar{N}_i} \left(\alpha_{kj} - \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} \right) - \sum_{k \in N_{ij}} \left(\sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh} - \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} \right) \right] \right\} \cdot e_i \\
&\quad - \frac{1}{2} \left\{ 1 - \sum_{k \in N_j \setminus \{i\}} \alpha_{jk} + \sum_{k \in N_{ij}} \beta_{ikj} + \frac{\rho}{1 + (d_{ij} + 1)\rho} \left[\sum_{k \in N_j \setminus \bar{N}_i} \left(\alpha_{kj} - \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} \right) \right. \right. \\
&\quad \left. \left. - \sum_{k \in N_i \setminus \bar{N}_j} \left(\alpha_{ki} - \sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh} \right) - \sum_{k \in N_{ij}} \left(\sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} - \sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh} \right) \right] \right\} \cdot e_j \\
&\quad + \frac{1}{2} \sum_{k \in N_{ij}} \left\{ \alpha_{ki} - \alpha_{kj} + \sum_{h \in N_{ijk}} (\beta_{ihk} - \beta_{jkh}) - \sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ihk} + \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} \right. \\
&\quad \left. + \frac{\rho}{1 + (d_{ij} + 1)\rho} \left[\sum_{k \in N_i \setminus \bar{N}_j} \left(\alpha_{ki} - \sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh} \right) - \sum_{k \in N_j \setminus \bar{N}_i} \left(\alpha_{kj} - \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} \right) \right. \right. \\
&\quad \left. \left. - \sum_{k \in N_{ij}} \left(\sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh} - \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} \right) \right] \right\} \cdot e_k + C_{ij}
\end{aligned}$$

The last equality is obtained by collecting terms and switching summand indexes. ■

B.2 Uniqueness in Minimally Connected Networks

Proposition 10. *Under the independent CARA-Normal setting, if the network is minimally connected, then there is a unique profile of transfer rules in \mathcal{T}^* that is Pareto efficient, and it takes the form of the local equal sharing rule.*

Proof. Consider minimally connected network G . For Pareto efficiency, we need for all linked pair ij

$$\frac{\mathbb{E}_{ij} [u'_i(x_i)]}{\mathbb{E}_{ij} [u'_j(x_j)]} = c_{ij}.$$

As the network is minimally connected, we have $N_{ij} = \emptyset$. Notice that

$$\mathbb{E} \left[r e^{-r(e_i - t_{ij} - \sum_{k \in N_i \setminus \{j\}} t_{ik}(e_i, e_k))} \middle| e_i, e_j \right] = c_{ij} \mathbb{E} \left[r e^{-r(e_j + t_{ij} - \sum_{h \in N_j \setminus \{i\}} t_{jh}(e_j, e_h))} \middle| e_i, e_j \right].$$

⁴²By independence,

$$\begin{aligned} & \mathbb{E} \left[r e^{-r(e_i - t_{ij} - \sum_{k \in N_i \setminus \{j\}} t_{ik}(e_i, e_k))} \middle| e_i \right] = c_{ij} \mathbb{E} \left[r e^{-r(e_j + t_{ij} - \sum_{h \in N_j \setminus \{i\}} t_{jh}(e_j, e_h))} \middle| e_j \right] \\ \Leftrightarrow & e^{-r(e_i - t_{ij})} \cdot \prod_{k \in N_i \setminus \{j\}} \mathbb{E} [e^{rt_{ik}(e_i, e_k)} | e_i] = c_{ij} e^{-r(e_j + t_{ij})} \cdot \prod_{h \in N_j \setminus \{i\}} \mathbb{E} [e^{rt_{jh}(e_j, e_h)} | e_j] \\ \Leftrightarrow & e_i - t_{ij} - \frac{1}{r} \sum_{k \in N_i \setminus \{j\}} \ln \mathbb{E} [e^{rt_{ik}(e_i, e_k)} | e_i] = e_j + t_{ij} - \frac{1}{r} \sum_{h \in N_j \setminus \{i\}} \ln \mathbb{E} [e^{rt_{jh}(e_j, e_h)} | e_j] - \frac{1}{r} \ln c_{ij} \\ \Leftrightarrow & t_{ij} = \frac{1}{2} e_i - \frac{1}{2} e_j - \frac{1}{2r} \sum_{k \in N_i \setminus \{j\}} \ln \mathbb{E} [e^{rt_{ik}(e_i, e_k)} | e_i] + \frac{1}{2r} \sum_{h \in N_j \setminus \{i\}} \ln \mathbb{E} [e^{rt_{jh}(e_j, e_h)} | e_j] + \frac{1}{2r} \ln c_{ij} \end{aligned} \tag{28}$$

$$= \frac{1}{2} e_i - \frac{1}{2} e_j - \frac{1}{2r} \sum_{k \in N_i \setminus \{j\}} \ln T_{ik} + \frac{1}{2r} \sum_{h \in N_j \setminus \{i\}} \ln T_{jh} + \frac{1}{2r} \ln c_{ij}$$

where

$$T_{ik} := \mathbb{E} [e^{rt_{ik}(e_i, e_k)} | e_i]$$

⁴²We hope the unfortunate notational coincidence of the endowment vector e and the natural exponential power e^{\cdot} will not result in any confusion.

Then, taking conditional expectations of (28), we have

$$\begin{aligned} T_{ij} &= e^{r\left(\frac{1}{2}e_i + \frac{1}{2}r\sigma^2 - \frac{1}{2r} \sum_{k \in N_i \setminus \{j\}} \ln T_{ik} + \frac{1}{2r} \ln \alpha_{ij}\right)} \cdot \mathbb{E} \left[e^{r \frac{1}{2r} \sum_{h \in N_j \setminus \{i\}} \ln T_{jh}} \middle| e_i \right] \\ &= e^{r\left(\frac{1}{2}e_i + \frac{1}{2}r\sigma^2 - \frac{1}{2r} \sum_{k \in N_i \setminus \{j\}} \ln T_{ik} + \frac{1}{2r} \ln \alpha_{ij}\right)} \cdot \prod_{h \in N_j \setminus \{i\}} \mathbb{E} \left[T_{jh}^{\frac{1}{2}} \right] \end{aligned}$$

and

$$\frac{1}{r} \ln T_{ij} = \frac{1}{2}e_i + \frac{1}{2}r\sigma^2 - \frac{1}{2} \sum_{k \in N_i \setminus \{j\}} \frac{1}{r} \ln T_{ik} + \frac{1}{2r} \ln \alpha_{ij} + \sum_{h \in N_j \setminus \{i\}} \ln \mathbb{E} \left[T_{jh}^{\frac{1}{2}} \right].$$

Introducing notation

$$\tilde{T}_{ij} = \frac{1}{r} \ln T_{ij},$$

we have

$$\tilde{T}_{ij} = \frac{1}{2}e_i - \frac{1}{2} \sum_{k \in N_i \setminus \{j\}} \tilde{T}_{ik} + c_{ij}$$

\Rightarrow

$$\sum_{j \in N_i} \tilde{T}_{ij} = \frac{d_i}{2}e_i - \frac{1}{2} \cdot (d_i - 1) \sum_{j \in N_i} \tilde{T}_{ik} + \sum_{j \in N_i} c_{ij}$$

\Rightarrow

$$\sum_{j \in N_i} \tilde{T}_{ij} = \frac{d_i}{d_i + 1}e_i + \frac{2}{d_i + 1} \sum_{j \in N_i} c_{ij}$$

\Rightarrow

$$\begin{aligned} \tilde{T}_{ij} &= \frac{1}{2}e_i - \frac{1}{2} \sum_{k \in N_i \setminus \{j\}} \tilde{T}_{ik} + c_{ij} \\ &= \frac{1}{2}e_i - \frac{1}{2} \sum_{k \in N_i} \tilde{T}_{ik} + \frac{1}{2}\tilde{T}_{ij} + c_{ij} \end{aligned}$$

\Rightarrow

$$\frac{1}{2}\tilde{T}_{ij} = \frac{1}{2} \left(e_i - \frac{d_i}{d_i + 1}e_i - \frac{2}{d_i + 1} \sum_{k \in N_i} c_{ik} \right) + c_{ij}$$

\Rightarrow

$$\tilde{T}_{ij} = \frac{1}{d_i + 1}e_i - \frac{1}{d_i + 1} \sum_{k \in N_i} c_{ik} + c_{ij}$$

Hence, by (28), we have

$$\begin{aligned}
t_{ij} &= \frac{1}{2}e_i - \frac{1}{2}e_j - \frac{1}{2} \sum_{k \in N_i \setminus \{j\}} \left(\frac{1}{d_i + 1}e_i - \frac{1}{d_i + 1} \sum_{k' \in N_i} c_{ik'} + c_{ik} \right) \\
&\quad + \frac{1}{2} \sum_{h \in N_j \setminus \{i\}} \left(\frac{1}{d_j + 1}e_j - \frac{1}{d_j + 1} \sum_{h' \in N_j} c_{jh'} + c_{jh} \right) + \frac{1}{2r} \ln \alpha_{ij} \\
&= \frac{1}{2} \left(1 - \frac{d_i - 1}{d_i + 1} \right) e_i - \frac{1}{2} \left(1 - \frac{d_i - 1}{d_i + 1} \right) e_j + C_{ij} \\
&= \frac{1}{d_i + 1} e_i - \frac{1}{d_i + 1} e_j + C_{ij}.
\end{aligned}$$

■

B.3 Linear Pareto Efficient Transfer Shares for Boundary Correlation Parameters

Proposition 11.

- For $\rho = -\frac{1}{n-1}$ and any network structure G such that $\max_{i \in N} d_i = n - 1$, let i^* be any individual with $d_{i^*} = n - 1$. Then a Pareto efficient profile of transfer rules is given by

$$\alpha_{ji^*} = 1, \quad \alpha_{i^*j} = \alpha_{jk} = 0, \quad \forall j, k \in N \setminus \{i^*\}.$$

- For $\rho = 1$ and any network structure G , any profile of transfer rules that satisfies the Kirchhoff Circuit Law as defined below is Pareto efficient:

$$\sum_{j \in \bar{N}_i} \alpha_{ij} = \sum_{j \in \bar{N}_i} \alpha_{ji} \quad \forall i \in N.$$

Proof. For $\rho = -\frac{1}{n-1}$ and G s.t. $\max_{i \in N} d_i = n - 1$, the profile of transfer rules given above attains zero variance in consumption for each individual, and is thus Pareto efficient. For $\rho = 1$, any profile of transfer rules that satisfies the Kirchhoff Circuit Law achieves the same profile of consumption plan as the null transfer (autarky), which is clearly Pareto efficient. ■

B.4. Detailed Specification and Proof for Proposition 8

Specifically, we assume that the correlation between e_i and e_j geometrically decays with social distance between i and j :

$$\text{Corr}(e_i, e_j) = \varrho^{\text{dist}(i,j)},$$

where the social distance $\text{dist}(i, j)$ is formally defined as the length (i.e., the number of links) of the shortest path connecting i and j in network G . For notational simplicity we set $\sigma^2 = 1$.

For tractability, we restrict attention to circle networks with $n = 2m + 1$ individuals. A n -circle consists of n individuals and n links: $G_{i,i+1} = 1$ for $i = 1, \dots, n$.⁴³ For any linked pair $i, i + 1$ along a n -circle (with $n \geq 4$), the conditional distribution of e_{i-1} (and similarly for e_{i+2}) is

$$e_{i-1}|_{e_i, e_{i+1}} \sim \mathcal{N}(\varrho e_i, 1 - \varrho).$$

Following a similar argument as in Section 4.2, we obtain the following condition for Pareto efficiency subject to local information constraints:

$$\begin{cases} \alpha_{i,i+1} &= \frac{1}{2} (1 - \alpha_{i,i-1} + \varrho \alpha_{i-1,i}) \\ \alpha_{i+1,i} &= \frac{1}{2} (1 - \alpha_{i+1,i+2} + \varrho \alpha_{i+2,i+1}) \end{cases}$$

for all $i \in N$. Then, the unique and symmetric solution for the above system is given by

$$\alpha_{ij}^* \equiv \alpha^{geo}(\varrho) = \frac{1}{3 - \varrho} \quad \forall G_{ij} = 1.$$

Under α^* , the final consumption for each individual is

$$x_i^{geo}(\varrho) = \frac{1}{3 - \varrho} e_{i-1} + \frac{1 - \varrho}{3 - \varrho} e_i + \frac{1}{3 - \varrho} e_{i+1}$$

⁴³We, for notational simplicity, define individual $n + 1$ to be individual 1, and individual 0 to be individual n .

with a variance of

$$Var_{geo,\varrho}(x_i^{geo}(\varrho)) = \frac{1 + \varrho}{3 - \varrho}.$$

In comparison, under the symmetric correlation structure in Section 4.2, the condition for Pareto efficiency on a n -circle is

$$\alpha_{i,i+1} = \frac{1}{2} \left[1 - \alpha_{i,i-1} + \frac{\rho}{1 + \rho} (\alpha_{i-1,i} - \alpha_{i+2,i+1}) \right]$$

with its unique and symmetric solution being

$$\alpha_{ij} \equiv \alpha^{unif}(\rho) = \frac{1}{3} \quad \forall G_{ij} = 1,$$

which is exactly the local equal sharing rule. This implies a final consumption of

$$x_i^{unif}(\rho) = \frac{1}{3}e_{i-1} + \frac{1}{3}e_i + \frac{1}{3}e_{i+1}$$

with a variance of

$$Var_{unif,\rho}(x_i^{unif}(\rho)) = \frac{1 + 2\rho}{3}.$$

We compare the correlation structures by setting ρ and ϱ to be such that each individual's consumption variance is equalized across the two correlation structures under the global equal sharing rule (which achieves first best risk sharing):

$$x_i^{FB} = \frac{1}{n} \sum_{k \in N} e_k.$$

The consumption variances that this sharing rule implies for the two correlation structures are:

$$\begin{aligned} Var_{unif,\rho}(x_i^{FB}) &= \frac{1 + 2m\rho}{2m + 1} \\ Var_{geo,\varrho}(x_i^{FB}) &= \frac{1 + 2 \sum_{k=1}^m \varrho^k}{2m + 1} = \frac{2 \frac{1 - \varrho^{m+1}}{1 - \varrho} - 1}{2m + 1}, \end{aligned}$$

The first-best total variances under the two correlations structures are equal if and

only if

$$\begin{aligned} Var_{unif,\rho}(x_i^{FB}) = Var_{geo,\varrho}(x_i^{FB}) &\Leftrightarrow \frac{1+2m\rho}{2m+1} = \frac{2\frac{1-\varrho^{m+1}}{1-\varrho} - 1}{2m+1} \\ &\Leftrightarrow \rho = \rho_m(\varrho) := \frac{\varrho(1-\varrho^m)}{m(1-\varrho)}. \end{aligned}$$

Noticing that the total variances without risk sharing at all are both equal to $(2m+1)$ under either correlation structure, setting $\rho = \rho_m(\varrho)$ implies that the total amount of sharable risk is equalized between the two correlation structures. Next we compare the consumption variances given Pareto efficient risk-sharing arrangements subject to local information constraints.

Notice that

$$Var_{unif,\rho}(x_i^{unif}(\rho)) \leq Var_{geo,\varrho}(x_i^{geo}(\varrho)) \Leftrightarrow \rho \leq \bar{\rho}(\varrho) := \frac{2\varrho}{3-\varrho}.$$

Hence, whenever

$$m > \frac{(3-\varrho)(1-\varrho^m)}{2(1-\varrho)}$$

we will have $\rho(\varrho) < \bar{\rho}(\varrho)$ and thus $Var_{uni}^\rho(x_i^{unif}(\rho)) < Var_{geo}^\varrho(x_i^{geo}(\varrho))$. In other words, fixing ϱ , efficient risk sharing subject to the local information constraint performs strictly better under the uniform correlation setting than under the geometrically decaying setting.

Moreover, the difference can be very stark. As $m \rightarrow \infty$,

$$\rho = \rho(\varrho) = \frac{\varrho(1-\varrho^m)}{m(1-\varrho)} \rightarrow 0,$$

and thus

$$Var_{unif,\rho}(x_i^{unif}(\rho)) = \frac{1+2\rho}{3} \rightarrow \frac{1}{3}, \quad \text{as } m \rightarrow \infty$$

while

$$Var_{geo,\rho}(x_i^{geo}(\varrho)) = \frac{1+\varrho}{3-\varrho} \quad \forall m.$$

When also taking $\varrho \rightarrow 1$ (after taking $m \rightarrow \infty$), we get

$$\lim_{\varrho \rightarrow 1} \lim_{m \rightarrow \infty} \text{Var}_{unif, \rho(\varrho)} \left(x_i^{unif}(\rho(\varrho)) \right) = \frac{1}{3},$$

$$\lim_{\varrho \rightarrow 1} \lim_{m \rightarrow \infty} \text{Var}_{geo, \rho} \left(x_i^{geo}(\varrho) \right) = 1.$$

B.5 Quadratic Utility Functions

With quadratic utility functions $u_i(x_i) = x_i - \frac{1}{2}rx_i^2$, the localized Borch rule requires that

$$\frac{\lambda_j}{\lambda_i} = \frac{\mathbb{E}_{ij} [u'_i(x_i)]}{\mathbb{E}_{ij} [u'_i(x_j)]} = \frac{\mathbb{E}_{ij} [1 - rx_i]}{\mathbb{E}_{ij} [1 - rx_j]}$$

$$\Leftrightarrow \lambda_i - \lambda_i r \left(\mu_i + e_i - t_{ij} - \sum_{h \in N_i} E_{ij} [t_{ih}] \right) = \lambda_j - \lambda_j \left(\mu_j + e_j + t_{ij} - \sum_{h \in N_j} \mathbb{E}_{ij} [t_{jh}] \right)$$

$$\Leftrightarrow r(\lambda_i + \lambda_j) t_{ij} = -(\lambda_i - \lambda_j) + \lambda_i r \left(\mu_i + e_i - \sum_{h \in N_i \setminus j} \mathbb{E}_{ij} [t_{ih}(I_{ih})] \right) - \lambda_j r \left(e_j - \sum_{h \in N_j \setminus i} \mathbb{E}_{ij} [t_{jh}(I_{jh})] \right)$$

$$\Leftrightarrow t_{ij} = \frac{\lambda_i}{\lambda_i + \lambda_j} \left(\mu_i + e_i - \sum_{h \in N_i \setminus j} \mathbb{E}_{ij} [t_{ih}(I_{ih})] \right) - \frac{\lambda_j}{\lambda_i + \lambda_j} \left(\mu_j + e_j - \sum_{h \in N_j \setminus i} \mathbb{E}_{ij} [t_{jh}(I_{jh})] \right) - \frac{\lambda_i - \lambda_j}{r(\lambda_i + \lambda_j)}$$

Postulating a bilateral linear rule:

$$t_{ij}(I_{ij}) = \alpha_{ij}e_i - \alpha_{ji}e_j + c_{ij}$$

Notice that this is equivalent to specifying $t_{ij}(I_{ij}) = \alpha_{ij}y_i - \alpha_{ji}y_j + c_{ij}$ as we allow μ_{ij} are simultaneously determined along with:

$$\begin{aligned} x_i &= \left(1 - \sum_{j \in N_i} \alpha_{ij}\right) e_i + \sum_{j \in N_i} \alpha_{ji}e_j + \mu_i - \sum_{j \in N_i} c_{ij} \\ &\equiv \left(1 - \sum_{j \in N_i} \alpha_{ij}\right) y_i + \sum_{j \in N_i} \alpha_{ji}y_j + \left(\sum_{j \in N_i} \alpha_{ji}\mu_i - \sum_{j \in N_i} \alpha_{ij}\mu_j\right) - \sum_{j \in N_i} c_{ij} \end{aligned}$$

Plugging in the postulation,

$$\begin{aligned} t_{ij} &= \frac{\lambda_i}{\lambda_i + \lambda_j} \left(\left(1 - \sum_{h \in N_i \setminus j} \alpha_{ih}\right) e_i + \sum_{h \in N_{ij}} \alpha_{hi}e_h + \sum_{h \in N_i \setminus \bar{N}_j} \alpha_{hi} \mathbb{E}_{ij}[e_h] \right) \\ &\quad - \frac{\lambda_j}{\lambda_i + \lambda_j} \left(\left(1 - \sum_{h \in N_j \setminus i} \alpha_{jh}\right) e_j + \sum_{h \in N_{ij}} \alpha_{hj}e_h + \sum_{h \in N_j \setminus \bar{N}_i} \alpha_{hi} \mathbb{E}_{ij}[e_h] \right) \\ &\quad + \frac{\lambda_i \left(\mu_i - \sum_{h \in N_i \setminus j} c_{ih}\right) - \lambda_j \left(\mu_j - \sum_{h \in N_j \setminus i} c_{jh}\right)}{\lambda_i + \lambda_j} - \frac{\lambda_i - \lambda_j}{r(\lambda_i + \lambda_j)} \\ &= \frac{\lambda_i}{\lambda_i + \lambda_j} \left(\left(1 - \sum_{h \in N_i \setminus j} \alpha_{ih}\right) e_i + \sum_{h \in N_{ij}} \alpha_{hi}e_h + \frac{\rho}{1 + (d_{ij} + 1)\rho} \sum_{h \in N_i \setminus \bar{N}_j} \alpha_{hi} \cdot \sum_{k \in \bar{N}_{ij}} e_k \right) \\ &\quad - \frac{\lambda_j}{\lambda_i + \lambda_j} \left(\left(1 - \sum_{h \in N_j \setminus i} \alpha_{jh}\right) e_j + \sum_{h \in N_{ij}} \alpha_{hj}e_h + \frac{\rho}{1 + (d_{ij} + 1)\rho} \sum_{h \in N_j \setminus \bar{N}_i} \alpha_{hi} \cdot \sum_{k \in \bar{N}_{ij}} e_k \right) \\ &\quad + \frac{\lambda_i \left(\mu_i - \sum_{h \in N_i \setminus j} c_{ih}\right) - \lambda_j \left(\mu_j - \sum_{h \in N_j \setminus i} c_{jh}\right)}{\lambda_i + \lambda_j} - \frac{\lambda_i - \lambda_j}{r(\lambda_i + \lambda_j)} \end{aligned}$$

In the special case of equal weighting: $\lambda_i = \lambda_j$, we have

$$\begin{aligned} \alpha_{ij} &= \frac{1}{2} \left(1 - \sum_{h \in N_i} \alpha_{ih} + \frac{\rho}{1 + (d_{ij} + 1)\rho} \left(\sum_{h \in N_i \setminus \bar{N}_j} \alpha_{hi} - \sum_{h \in N_j \setminus \bar{N}_i} \alpha_{hi} \right) \right) \\ c_{ij} &= \frac{1}{2} (\mu_i - \mu_j) - \frac{1}{2} \left(\sum_{h \in N_i \setminus j} c_{ih} - \sum_{h \in N_j \setminus i} c_{jh} \right) \end{aligned}$$

Note that system of linear equations in α is exactly the same one as in Section 4.

B.6 Detailed Specification and Proof for Proposition 9

In our setting, for a given network G , individual i 's Myerson value is defined by

$$MV_i(G) := \sum_{S \subseteq N} \frac{(\#(S) - 1)(n - \#(S))}{n!} \cdot \frac{1}{2} r \sigma^2 \left[TVar \left(G|_{(S \setminus \{i\})} \right) + \sigma^2 - TVar \left(G|_S \right) \right]$$

where $\#(S)$ denotes the number of individuals in a subset S of N , and $G|_S$ denotes the subgraph of G restricted to the subset S of individuals. Given the CARA-normal specification, $TVar \left(G|_{(S \setminus \{i\})} \right) + \sigma^2 - TVar \left(G|_S \right)$ is the surplus from risk reduction through i 's links in S .

Notice that, given any $S \subseteq N$,

$$TVar \left(G|_{S \setminus \{i\}} \right) - TVar \left(G|_S \right) = 1 - \frac{1}{d_i(G|_S) + 1} + \sum_{k \in N_i(G|_S)} \frac{1}{d_k(G|_S) [d_k(G|_S) + 1]},$$

which is strictly increasing in $d_i(G|_S)$ but strictly decreasing in $d_k(G|_S)$ for each $j \in N_k(G|_S)$. Moreover, for any $k \in N$, $d_k(G|_S)$ is weakly increasing in S , i.e., $S \subseteq S' \Rightarrow d_k(G|_S) \leq d_k(G|_{S'})$.

Consider any pairwise stable network G under the Myerson-value transfers. Then, if i, j are linked, it must be that

$$MV_i(G) - MV_i(G - ij) \geq c.$$

Fixing ij , for each $S \subseteq N$, we have

$$\begin{aligned} & TVar \left(G - ij|_{S \setminus \{i\}} \right) - TVar \left(G - ij|_S \right) \\ = & \begin{cases} TVar \left(G|_{S \setminus \{i\}} \right) - TVar \left(G|_S \right), & \text{if } j \notin S \\ 1 - \frac{1}{d_i(G|_S)} + \sum_{k \in N_i(G|_S) \setminus \{j\}} \frac{1}{d_k(G|_S) [d_k(G|_S) + 1]}, & \text{if } j \in S \end{cases} \end{aligned}$$

so

$$\begin{aligned}
& \left[TVar \left(G|_{S \setminus \{i\}} \right) - TVar \left(G|_S \right) \right] - \left[TVar \left(G - ij|_{S \setminus \{i\}} \right) - TVar \left(G - ij|_S \right) \right] \\
&= \mathbb{1} \{j \in S\} \cdot \left[\frac{1}{d_i(G|_S) [d_i(G|_S) + 1]} + \frac{1}{d_j(G|_S) [d_j(G|_S) + 1]} \right] \\
&\geq \mathbb{1} \{j \in S\} \cdot \left[\frac{1}{d_i(G) [d_i(G) + 1]} + \frac{1}{d_j(G) [d_j(G) + 1]} \right]
\end{aligned}$$

Averaging over all possible $S \subseteq N$, we get

$$MV_i(G) - MV_i(G - ij) \geq \frac{1}{2} \cdot \left[\frac{1}{d_i(G) [d_i(G) + 1]} + \frac{1}{d_j(G) [d_j(G) + 1]} \right]$$

as

$$\sum_{S \subseteq N} \frac{(\#(S) - 1)(n - \#(S))}{n!} \mathbb{1} \{j \in S\} = Pr \{i \text{ arrives later than } j\} = \frac{1}{2}.$$

From the perspective of social efficiency, the link ij in G is (strictly) socially efficient if

$$\frac{1}{d_i(G) [d_i(G) + 1]} + \frac{1}{d_j(G) [d_j(G) + 1]} > 2c.$$

Thus we can conclude that, given any pairwise stable network G under the Myerson-value transfers, whenever a link ij is (strictly) socially efficient, it will be present in G , because the increments in both i 's and j 's private benefits strictly exceed the cost of linking c :

$$\begin{aligned}
MV_i(G) - MV_i(G - ij) &> \frac{1}{2} \cdot 2c = c \\
MV_j(G) - MV_j(G - ij) &> \frac{1}{2} \cdot 2c = c.
\end{aligned}$$